

Preface

This university textbook is intended for the use by teaching the subject Applied Informatics in the first year of engineering study of the Faculty of Electrical Engineering and Informatics, Technical University.

The textbook follows the content of mathematical subjects in bachelor degree grade. The content of the textbook is intended by the curriculum of the subject, that have been drafted according to the requirements specialized departments. The aim is to give the basic methods necessary for further study of technical subjects.

For more illustrative explanation of each method are solved simple examples. In addition, at the end of each chapter are given tasks for individual study. A suitable complement to practice materials are also other materials available on the website of Department of Mathematics and Theoretical Informatics Faculty of Electrical Engineering and Informatics Technical University in Košice <http://www.tuke.sk/fei-km/> in the part Predmety/Výučba.

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This textbook is available on CD and on the web site DMTI FEEI TUKE (KMTI FEEI TU) and Moodle system, which is managed by the FEEI TUKE.

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Authors

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List of Abbreviations and Symbols

$F(x)$ – distribution function

$f(x)$ – density

$\binom{n}{m}$ – n choose m , binomial number

$n!$ – n factorial

$P(n)$ – permutation of n different things

$P'_{n_1, \dots, n_c}(n)$ – permutations with repetition

$C(n, k)$ – combinations

$C'(n, k)$ – combinations with repetitions

$V(n, k)$ – variations

$V'(n, k)$ – variations with repetitions

A – event

$P(A)$ – probability of an event

$P(A|B)$ – conditional probability of A given B

$X(E)$ – a value of a random variable X for the event E

$P(X = a)$ – the probability that the value of the random variable X is equal to the number a

$E(X)$ – an expected value of random variable X (central tendency)

$D(X)$ – a variance of random variable X (dispersion)

$\sigma(X)$ – a standard deviation of random variable X is $\sqrt{D(X)}$

x_i – i -th value of the measurement

$x_{(i)}$ – ordered value of the non-decreasing sequence of values on the i -th position

n – size of population

m – number of classes

n_i – absolute class frequency

f_i – relative class frequency

N_i – absolute cumulative class frequency

F_i – cumulative relative class frequency

R – range

\bar{x} – mean

\bar{x}_G – geometric mean

\bar{x}_H – harmonic mean

\tilde{x} – median

\hat{x} – mode

s^2 – variance

s – standard deviation

s_{n-1}^2 – sample variation

s_{n-1} – sample standard deviation

γ_3 – (sample) skewness

γ_4 – (sample) kurtosis

Chapter 1

Introduction to Probability Theory

1.1 Permutations, Variations, and Combinations

Combinatorics has many applications within computer science for solving complex problems. However, it is under-represented in literature since there is little application of combinatorics in business applications. Fortunately, the science behind it has been studied by mathematicians for centuries, and is well understood and well documented. However, mathematicians are focused on in how many elements will exist within a combinatorics problem, and have little interest in actually going through the work of creating those lists. There are two common combinatorial concepts that are taught in every probability course. These are permutations and combinations. There is a smaller group known as variation, which combines the properties of permutations and combinations. In addition, there are variants of each of the three combinatorial collections which include the introduction of repetitions to input or output. So, the complete list of combinatorial collections is:

- permutations,
- permutations with repetitions,
- combinations,
- combinations with repetitions,
- variations,
- variations with repetitions.

Permutations deal with organizing a set of items, for example, how many ways a deck of 52 cards can be shuffled. Combinations deal with subsets of a set of items, for example, in how many ways a poker card can be dealt into 5 hands from a deck of 52 cards. In both cases, each card in the deck or hand is unique, so there is no repetition. However, problems occur when repetition occurs at the input and/ or output. In these cases, the repeating versions allow us more options when compiling our output files. Variations are used when not only the subgroup provided by the combinations is relevant, but also the

arrangement within that subgroup. In the following, we shall need the *factorial function* defined as follows

$$0! = 1, \quad (n + 1)! = (n + 1)n! \quad (1.1)$$

and the *binomial coefficients*, defined by the formula

$$\binom{n}{k} = \frac{n!}{(n - k)! \cdot k!}, \quad (n, k \text{ are nonnegative integers, } n \geq k) \quad (1.2)$$

From (1.1) and (1.2), we obtain

$$\binom{n}{0} = 1, \quad \text{in particular,} \quad \binom{0}{0} = 1.$$

Permutations

Permutations are all possible orderings of a given input set. Each ordering of the input is called a permutation. When each item in the input set is different, there is only one way to generate the permutations. However, when two or more items in the set are the same, two different permutation sets are possible. These are called Permutations and Permutations with Repetition.

Theorem 1.1 (Permutations). The number of permutations of n different items taken all at a time is

$$P(n) = n! = 1 \cdot 2 \cdot \dots \cdot n. \quad (1.3)$$

Permutations with Repetition

Permutations with Repetition sets give allowance for repetitive items in the input set that reduce the numbers of orderings. The number of Permutations with Repetition is not as large, being reduced by the number and count of repetitive items in the input set.

Theorem 1.2 (Permutations with repetition). If n given items can be divided into c classes of alike items differing from class to class, then the number of permutations of these items taken all at a time is

$$P'_{n_1, n_2, \dots, n_c}(n) = \frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_c!} \quad (n_1 + n_2 + \dots + n_c = n) \quad (1.4)$$

where n_j is the number of items in the j th class.

Combinations

Combinations are subsets of a given size taken from a given input set. The size of the set is known as the Upper Index (n) and the size of the subset is known as the Lower

Index (k). Unlike permutations, combinations do not have any order in the output set. Like permutations, they do have two generation methods based on the repeating of output items. These are called Combinations and Combinations with Repetition.

Combinations can be thought of as throwing a set of n dominos into a hat and then retrieving k of them. Each domino can only be chosen once, and the order that they were chosen out of the hat is irrelevant.

Theorem 1.3 (Combinations). The number of different combinations of n different items, k at time is

$$C(n, k) = \binom{n}{k}. \quad (1.5)$$

Combinations with Repetitions

Combinations with Repetition are determined by looking at a set of items, and selecting a subset while allowing repetition. For example, choose a tile from the scrabble bag above, write down the letter, and return the letter to the bag.

Theorem 1.4 (Combinations with repetitions). The number of different combinations of n different items, k at time with repetitions is

$$C'(n, k) = \binom{n + k - 1}{k}. \quad (1.6)$$

Variations

Variations combine features of combinations and permutations, they are the set of all ordered combinations of items to make up a subset. Like combinations, the size of the set is known as the Upper Index (n) and the size of the subset is known as the Lower Index (k). The generation of variations can be based on the repeating of output items. These are called Variations and Variations with Repetition.

Variations are permutations of combinations. That is, a variation of a set of n items choose k , is the ordered subset of size k .

Theorem 1.5 (Variations). The number of different variations of n different items, taken k at time is

$$V(n, k) = \frac{n!}{(n - k)!}. \quad (1.7)$$

Theorem 1.6 (Variations with repetitions). The number of different variations of n different items, taken k at time with repetitions is

$$V'(n, k) = n^k. \quad (1.8)$$

1.2 Combinatorial principles

We shall use two combinatorial principles.

The principle of multiplication

If the activity consists in k successive steps and the first step may be performed n_1 ways, the second step may be performed n_2 ways,
 \vdots
 the k -th step may be performed n_k ways,
 then number of different ways of performing an action is $n_1 \cdot n_2 \dots n_k$.

The principle of addition

Let us have the sets A_1, A_2, \dots, A_k that are pairwise disjoint. Let $|A_i| = n_i$. Then $|A_1 \cup A_2 \cup \dots \cup A_k| = n_1 + n_2 + \dots + n_k$.

1.3 Classical Definition of Probability

The probability theory has the purpose of providing mathematical models of situations affected or governed by “chance effect” for instance, in weather forecasting, life insurance, quality of technical products, traffic problems, and, of course, games of chance with cards or dice.

We begin with defining some standard terms. An *experiment* is a process of measurement or observation, in a laboratory, in a factory, on the street, in nature, or wherever: so “experiment ” is used in rather general case. Our interest is in experiments that involve *randomness*, chance effects, so that we cannot predict a result exactly. A *trial* is a single performance of an experiment. Its result is called a *random event*. We shall use capital letters to denote events, for example A, B, C, \dots and write them as sets.

In rolling die, events are for example $A = \{1, 3, 5\}$ (“odd number”), $B = \{2, 4, 6\}$ (“even number”), $C = \{5, 6\}$ (“number greater than 4”), etc. Event that after the experiment will never be is called an *impossible event*, we write \emptyset . Event that after the experiment will always occur is called a *certain event*, we write I .

Basic knowledge of events

Definition 1.1.

1. Event A is a subset of event B , if it follows from the occurrence of event A , that event B has occurred.
2. Events A and B are *equivalent*, if event A occurs if and only if event B occurs, we write $A = B$.

3. The *complement* of A is an event, which occurs if and only if event A does not occur, we write \bar{A} .
4. The *union* of events A and B is an event, which occurs if occurs at least one of events A and B , we write $A \cup B$.
5. The *intersection* of events A and B is an event, which occurs if occur both of events A and B , we write $A \cap B$.

Throwing a dice once, let event A be that the number on the face is odd and event B be that the number on the face is greater than 4. We denote events as the sets. We have $A = \{1, 3, 5\}$ and $B = \{5, 6\}$. Then $\bar{A} = \{2, 4, 6\}$, $\bar{B} = \{1, 2, 3, 4\}$, $A \cap B = \{5\}$, $A \cup B = \{1, 3, 5, 6\}$, $A - B = \{1, 3\}$, $B - A = \{6\}$.

Definition 1.2.

1. Events A and B are called *disjoint* if they can not occur at the same time, i.e., $A \cap B = \emptyset$.
2. Events H_1, H_2, \dots, H_n are disjoint, if $H_i \cap H_j = \emptyset$ for each $i \neq j$, $i, j \in \{1, 2, \dots, n\}$.

Definition 1.3. The system of *disjoint* events H_1, H_2, \dots, H_n is called *complete*, if its union is a certain event, i.e., $\bigcup_{i=1}^n H_i = I$. The complete system of disjoint events is called the system of *hypothesis*.

When throwing dice, let H_1 denotes the event that the face shows an even number and H_2 denotes the event that the face shows an odd number. The events H_1, H_2 represent a complete system of disjoint events (hypothesis).

Definition 1.4.

1. An event E is called *simple*, if there does not exist events A_1, A_2 different from E such that $E = A_1 \cup A_2$.
2. Each event, which is not elementary, is called a *composite event*.
3. The set of all elementary events which can occur as the output of a random experiment is called a *space of elementary events*, i.e. $\gamma = \{E_1, E_2, \dots, E_n\}$.

When throwing dice, $\gamma = \{E_1, E_2, \dots, E_6\}$, where E_k denotes the event that the face shows the number k . An event A can be equivalent to some of elementary events but also can include several elementary events. For example, showing an odd number on the face consists of three elementary events E_1, E_2, E_3 .

The operations with random events are reduced to the operations with sets and they follow the same rules. The empty set corresponds to the impossible event and the space of elementary events corresponds to the certain event.

Definition 1.5. Let $\gamma = \{E_1, E_2, \dots, E_n\}$ be a space of elementary events. A nonempty system τ of subsets of γ is called a *sample space*, if

1. $\emptyset \in \tau$;
2. if $A, B \in \tau$ then $A \cap B \in \tau$, $A \cup B \in \tau$, $\bar{A} \in \tau$;
3. if $A_1, A_2, \dots, A_n \in \tau$ then $\bigcap_{i=1}^{\infty} A_i \in \tau$ and $\bigcup_{i=1}^{\infty} A_i \in \tau$.

Elements of a sample space are *events*.

Basic properties of operations with events:

1. $A \subset A$, $A \cup A = A$, $A \cap A = A$;
2. $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$.
3. $A \cup I = I$, $A \cap I = A$.
4. $A \cup \bar{A} = I$, $A \cap \bar{A} = \emptyset$.
5. $A \cup B = B \cup A$, $A \cap B = B \cap A$.
6. $A \cup (B \cap C) = (A \cup B) \cap C$, $A \cap (B \cup C) = (A \cap B) \cup C$.
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
8. $\overline{A \cup B} = \bar{A} \cap \bar{B}$, $\overline{A \cap B} = \bar{A} \cup \bar{B}$. (de Morgan rules)

Definition 1.6 (Classical definition of probability). Let $\gamma = \{E_1, E_2, \dots, E_n\}$ be a space of elementary events. Let each elementary event be “equally possible”. For each even $A \in \tau$ we define its *probability* as follows:

$$P(A) = \frac{m}{n}, \quad (1.9)$$

where m is the number of elementary events from which A consists, i. e. $A = E_{i_1} \cup E_{i_2} \cup \dots \cup E_{i_m}$.

We can equality (1.9) interpret as follows:

$$P(A) = \frac{\text{the number of favorable outcomes of event } A}{\text{the number of possible outcomes of event } A}.$$

Theorem 1.7 (Basic properties of classical probability).

1. For each $A \in \tau$ is $0 \leq P(A) \leq 1$.
2. $P(\emptyset) = 0$, $P(I) = 1$.
3. $P(\bar{A}) = 1 - P(A)$.
4. If $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$.

1.4 Solved Examples

Example 1.1. There is a basket of fruits containing an apple, a banana and an orange. Five girls have a taste for some fruit. In how many ways we can give to three of the five girls one fruit to eat and leave two of them without a fruit ?

Solution:

Giving the 3 fruits to 3 of the 5 girls is a sequential problem. We first give the apple to one of the girls. There are 5 possible ways to do this. Then we give the banana to one of the remaining girls. There are 4 possible ways to do this, because one girl has already been given a fruit. Finally, we give the orange to one of the remaining girls. There are 3 possible ways to do this, because two girls have already been given a fruit. Using the principle of multiplication we obtain that there are $5 \cdot 4 \cdot 3 = 60$ ways to give them fruits.

In fact, the number of ways to assign the three fruits is equal to the number of 3-variations of 5 objects (without repetition). If we denote it by $V(5, 3)$, then we get

$$V(5, 3) = \frac{5!}{(5-3)!} = 60. \quad \checkmark$$

Example 1.2. In a race with 30 runners 8 trophies will be given to the top 8 runners (the trophies are distinct: first place, second place, etc), in how many ways the trophies can be given away?

Solution:

This is a permutation problem since the trophies are distinct. Think of the trophies as being 8 positions. The number of ways to arrange 30 items taken 8 at a time is

$$V(30, 8) = \frac{30!}{(30-8)!} = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23.$$

This result can also be obtained by multiplying the number of first place options by the second, third place, etc. ✓

Example 1.3. How does change the solution of the problem in Example (3.1) if a certain person, Roberta, must be one of the top 3 winners?

Solution:

There are 3 ways to put Roberta in one of the top 3 positions. The number of ways to give other 7 trophies equals $V(29, 7)$. Using the principle of multiplication we obtain that the number of ways is

$$3 \cdot V(29, 7) = 3 \cdot (29 \cdot 28 \cdot 27 \cdot \dots \cdot 23). \quad \checkmark$$

Example 1.4. In how many ways can you choose 4 groups of 4 people from 16 people, assuming the groups are distinct?

Solution:

Choosing the members of the first group we choose 4 people from 16, which are combinations $C(16, 4)$. The members of the second and the third group are chosen from 12 and 8

people, respectively.

The number of ways, we can choose 4 groups of 4 people from 16 people is

$$\binom{16}{4} \cdot \binom{12}{4} \cdot \binom{8}{4} = 63063000. \quad \checkmark$$

Example 1.5. In how many ways can you pair up 8 boys and 8 girls?

Solution:

If sounds like there are 2 sets of 8 items to consider permuting, but in reality we are only permuting one set of 8 items. Think of the boys as in a fixed order: boy 1, boy 2, ..., boy 8. Each arrangement of girls corresponds to one pairing with the boys: girl 1 in the arrangement with boy 1, girl 2 in the arrangement with boy 2, etc. The girls can be arranged in $P(8) = 8!$ ways. ✓

Example 1.6. Ternary strings have symbols 0, 1, and 2. In how many ternary strings of length 4 there is exactly one 1?

Solution:

If there is exactly one 1, then there are 4 positions for the 1. The number of ways to fill the other 3 blanks with the 0 or the 2 equals the number of different variations of 2 elements taken 3 at time, so we have variations with repetition. Usind the principle of multiplication, the number of ternary strings of length 4 with exactly one 1 is

$$4 \cdot V'(2, 3) = 4 \cdot 2^3 = 32. \quad \checkmark$$

Example 1.7. In how many ways can you arrange a) 5 people b) 4 people on a ferris wheel with 6 seats, if the seats are indistinguishable?

Solution:

a) There are $(n - 1)!$ circular permutations of n items, since the permutation a_1, a_2, \dots, a_n is the same as a_2, \dots, a_n, a_1 , etc.. In our example, we have a circular permutation of essentially 6 items - 5 people and one empty seat. So the number of ways is 5!.

If there were 4 people and 2 empty seats, the answer would not be 5! since the 4 people are different, but the 2 empty seats are indistinguishable. For example, let a, b, c, d be the 4 people and e_1, e_2 be the empty seats. There is no real difference between a, b, c, d, e_1, e_2 and a, b, c, d, e_2, e_1 , but if you gave the answer $(6 - 1)! = 5!$ you would count these identical permutations twice. The answer for this situation would be $5!/2! = 5 \cdot 4 \cdot 3 = 60$.

✓

Example 1.8. How many words, including non-semantic ones, can be obtained by changing the order of the letters in the word“engineering”?

Solution:

Since it depends on the order and the letters are repeated, it is variations with repetition. the number of repetitions of letters e, n, g, i, r is $n_1 = 3, n_2 = 3, n_3 = 2, n_4 = 2, n_5 = 1$. Using formula (1.4) we obtain the number of words equal to

$$P'_{3,3,2,2,1}(11) = \frac{11!}{3! \cdot 3! \cdot 2! \cdot 2! \cdot 1!} \quad \checkmark$$

Example 1.9. A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?

Solution:

We have in total 20 marbles. Between them, 10 marbles are white. We have $n = 20$ and $m = 10$. According to (1.9), the probability that the drawn a white marble will be white is

$$P(A) = \frac{m}{n} = \frac{10}{20} = 0.5.$$

✓

Example 1.10. The blood groups of 200 people is distributed as follows: 50 people have type A blood, 65 people have B blood type, 70 people have 0 blood type and 15 people have AB blood type. If a person from this group is selected at random, what is the probability that this person has 0 blood type or AB?

Solution:

Let A be the event that We denote by letter A the event that a person has a blood group 0 and by letter B the event that a person has a blood group AB. Since events A , B are disjoint, according to Theorem 1.7 we have

$$P(A \cup B) = P(A) + P(B) = \frac{70}{200} + \frac{15}{200} = 0,425.$$

✓

Example 1.11. Suppose a coin is flipped 3 times. What is the probability of getting two tails and one head? Note that each flipping of coin has two possible outcomes H (heads) and T (Tails).

Solution:

For this experiment, the space of elementary events consists of 8 events:

$$\gamma = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$$

Each event is equally likely to occur. The event A “getting two tails and one head” consists of the following elementary events: $A = \{TTH, THT, HTT\}$.

Since $m = 3$ and $n = 8$, the probability of event A is

$$P(A) = \frac{3}{8}.$$

Note that the task can be solved without naming elements of γ and A . In fact, $n = \sqrt{V'(2, 3)} = 2^3 = 8$ and $m = C(3, 2) = 3$.

Example 1.12. Two dice are rolled, find the probability that the sum of obtained numbers is

- a) equal to 1,
- b) equal to 4,
- c) less than 13.

Solution:

The space of elementary events consists of all ordered pairs of numbers $1, 2, \dots, 6$, so there are $n = 6^2 = 36$ elementary events.

- a) Let A be the event “the sum equals 1”. Since there are no outcomes which correspond to a sum equal to 1, event A is an impossible event. Hence $P(A) = 0$.
- b) Let B be the event “the sum equals 4”. Three possible outcomes give a sum equal to 4: $B = \{(1, 3), (2, 2), (3, 1)\}$. We have $m = 3$, $n = 36$. Hence, $P(B) = \frac{3}{36} = \frac{1}{12}$.
- c) Let C be the event “the sum is less than 13”. All possible outcomes are together favorable, so $C = \gamma$. Hence, $P(C) = 1$. Event C is a certain event. ✓

Example 1.13. A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

Solution:

The space of elementary events of the experiment is

$$\gamma = \{(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (1, T), (2, T), (3, T), (4, T), (5, T), (6, T)\}.$$

Let A be the event “the die shows an odd number and the coin shows a head”. Event A may be described as $A = \{(1, H), (3, H), (5, H)\}$. The probability $P(A)$ is given by

$$P(A) = \frac{3}{12} = \frac{1}{4}. \quad \checkmark$$

1.4 Unsolved Tasks

1.1. There are 5 seats around a table and 5 people to be seated at the table. In how many ways can they seat themselves? [24]

1.2. Bob, John, Luke, and Tim play a tennis tournament. The rules of the tournament are such that at the end of the tournament a ranking will be made and there will be no ties. In how many different rankings can be there? [6]

1.3. A byte is a number consisting of 8 digits that can be equal either to 0 or to 1. In how many different bytes are there? [256]

1.4. An hexadecimal number is a number whose digits can take sixteen different values: either one of the ten numbers from 0 to 9, or one of the six letters from A to F. In how many different 8-digit hexadecimal numbers are there, if an hexadecimal number is allowed to begin with any number of zeros? [16⁸]

1.5. Three cards are drawn from a standard deck of 52 cards. In how many different 3-card hands can possibly be drawn?

1.6. John has got 1 dollar, with which he can buy green, red, and yellow candies. Each candy costs 20 cents. John will spend all the money he has on candies. In how many different combinations of green, red, and yellow candies can he buy? [21]

1.7. The board of directors of a corporation comprises 10 members. An executive board, formed by 4 directors, needs to be elected. In how many possible ways are there to form the executive board? [210]

1.8. John has a basket of fruits containing one apple, one banana, one orange, and one kiwi. He wants to give one fruit to each of his two little sisters and two fruits to his big brother. In in how many different ways can he do this? [12]

1.9. There are 12 teams in a soccer league, and each team must play with each other twice in a tournament. What is the number of games that will be played in total? [132]

1.10. What is the number of distinguishable arrangements that can be made from the word KITCHEN, if the vowels must stay together? [1440]

1.11. A family is being arranged in a line for a group photograph. If the family consists of a mother, a father, and five children, what is the number of arrangements that begin and end with a parent? [240]

1.12. There are ten people available for appointment to a committee consisting of six people. What is the number of committees that can be formed, if Kirsten and James must be in the committee? [70]

1.13. In how many ways can be drawn numbers in the lottery “5 from 40”? [658008]

1.14. What is the number of committees consisting of 4 men and 5 women that can be formed from 10 men and 13 women? [270270]

1.15. If all of the letters in the word PENCILS are used, what is the number of arrangements with all the consonants together? [720]

1.16. If repeated digits are not allowed, what is the number of three digit or four digit even numbers that can be formed from the numbers 2, 3, 5, 6, 7? [72]

1.17. The map of a small town has streets drawn vertically, and avenues drawn horizontally. A student wishes to walk to the recreation centre, which is 4 blocks East and 5 blocks South of his home. What is the number of different routes to the recreation center that are 9 blocks in length? [126]

1.18. A committee requires one accountant, two marketing agents, and four board members. What is the number of committees that can be formed, if there are four accountants, three marketing agents, and seven board members available for selection for the committee? [420]

1.19. There are 12 people in line for a movie. If Crystal, Steven, and Jason are friends and will always stand together, what is the total number of possible arrangements for the entire line? [21772800]

1.20. A security code used to consist of two odd digits, followed by four even digits. To allow more codes to be generated, a new system uses two even digits, followed by any three digits. If repeated digits are allowed, what is the increase in the number of security codes? [9375]

1.21. A multiple choice test has 15 questions. Tested candidates know that four of these questions have A as an answer, three have B as an answer, six have C as an answer, and two have D as an answer. What is the number of different answer sheets that can be created? [6306300]

1.22. Six points are drawn on a circle. What is the number of triangles that can be formed from these six points? [20]

1.23. There are 6 men and 9 women available for selection of a 6-person committee. If the committee must have at least one man, what is the number of possible committees? [4921]

1.24. One bag contains 5 coloured marbles and another bag contains 4 coloured marbles. None of the 9 marbles are of the same colour. If a person reaches into the first bag and pulls out two marbles, then reaches into the second bag and pulls out two marbles, what is the number of possible colour combinations? [60]

1.25. A student has 8 tiles that spell the word COMPUTER. If the student now wishes to use some of these tiles to make a four-letter word that contains exactly 2 vowels and exactly 2 consonants, what is the number of possible words? [720]

1.26. A boy going on a trip is told that out of his 8 favourite toys, he can bring at most three toys. What is the number of ways he could select which toys he brings? [56]

- 1.27.** A research team of 6 people is to be formed from 10 chemists, 5 politicians, 8 economists, and 15 biologists. What is the number of possible teams that can be formed with at least 5 chemists? [7266]
- 1.28.** You roll two dice. The first die shows a ONE and the other die rolls under the table and you cannot see it. Now, what is the probability that both die show ONE? $[\frac{1}{6}]$
- 1.29.** What is the probability that the sum of numbers on two dice will be greater than 8, given that the first die is 6? $[\frac{2}{3}]$
- 1.30.** Two fair six-sided dice are rolled and the face values are added. What is the probability of obtaining an odd number greater than 8? $[\frac{1}{6}]$
- 1.31.** Three cards are pulled from a deck of 52 cards. What is the probability of obtaining at least one club? [0.58647]
- 1.32.** Three different DVDs and their corresponding DVD cases are randomly strewn about on a shelf. If a young child puts the DVDs in the cases at random, what is the probability of correctly matching all DVDs and cases? $[\frac{1}{6}]$
- 1.33.** A 5 digit PIN number can begin with any digit (except for zero) and the remaining digits have no restriction. If repeated digits are allowed, what is the probability of the PIN code beginning with a 7 and ending with an 8? $[\frac{1}{90}]$
- 1.34.** On the shelf are laid 20 various books among which are 4 books about computers. What is the probability that these 4 books are placed side by side? $[\frac{1}{285}]$
- 1.35.** Tickets numbered from 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5? $[\frac{9}{20}]$
- 1.36.** A bag contains 2 red, 3 green, and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue? $[\frac{10}{21}]$
- 1.37.** In a box, there are 8 red, 7 blue, and 6 green balls. One ball is picked up randomly. What is the probability that it is neither red nor green? $[\frac{1}{3}]$
- 1.38.** Three unbiased coins are tossed. What is the probability of getting at most two heads? $[\frac{7}{8}]$

- 1.39.** Two dice are thrown simultaneously. What is the probability of getting two numbers whose product is even? $[\frac{3}{4}]$
- 1.40.** In a class, there are 15 boys and 10 girls. Three students are selected at random. What is the probability that 1 girl and 2 boys are selected? $[0.45652]$
- 1.41.** In a lottery, there are 10 prizes and 25 blanks. A lottery is drawn at random. What is the probability of getting a prize? $[0.28571]$
- 1.42.** From a pack of 52 cards, two cards are drawn together at random. What is the probability of both the cards being kings? $[0.00452]$
- 1.43.** Two dice are tossed. What is the probability that the sum of showed numbers is a prime number? $[\frac{7}{18}]$
- 1.44.** A card is drawn from a pack of 52 cards. What is the probability of getting a queen of club or a king of heart? $[\frac{1}{26}]$
- 1.45.** A bag contains 4 white, 5 red, and 6 blue balls. Three balls are drawn at random from the bag. What is the probability that all of them are red? $[0.02198]$
- 1.46.** Two cards are drawn together from a pack of 52 cards. What is the probability that one is a spade and one is a heart? $[0.12745]$
- 1.47.** One card is drawn at random from a pack of 52 cards. What is the probability that the card drawn is a face card (Jack, Queen, or King)? $[\frac{3}{13}]$
- 1.48.** A new bag of golf tees contains 10 red tees, 10 orange tees, 10 green tees, and 10 blue tees. You empty the tees into your golf bag. What is the probability of grabbing out two tees of the same colour in a row for you and your partner? $[0.23077]$
- 1.49.** In a library box, there are 8 novels, 8 biographies, and 8 war history books. If Jack selects two books at random, what is the probability of selecting two different kinds of books in a row? $[0.69565]$
- 1.50.** A survey of height school students asked: What is your favourite winter sport? The results are summarized below:

Grade	Snowboarding	Skiing	Ice Skating	TOTAL
6th	68	41	46	155
7th	84	56	70	210
8th	59	74	47	180
TOTAL	211	171	163	545

Using these 545 students as the sample space, a student from this study is randomly selected.

- a) What is the probability of selecting a student whose favourite sport is skiing?[0.31376]
- b) What is the probability of selecting a 6th grade student? [0.2844]
- c) If the selected student is a 7th grade student, what is the probability that the student prefers ice skating? [0.33333]
- d) If the student selected prefers snowboarding, what is the probability that the student is a 6th grade student? [0.32227]
- e) If the selected student is an 8th grade student, what is the probability that the student prefers skiing or ice skating? [0.67222]
- 1.51.** In a shipment of 20 computers, 3 are defective. Three computers are randomly selected and tested. What is the probability that all three are defective if the first and the second ones are not replaced after being tested? [0.000877]
- 1.52.** On a math test, 5 of 20 students got an A. If three students are chosen at random without replacement, what is the probability that all three got an A on the test? [0.00877]
- 1.53.** If a five card hand is dealt from a deck of 52 cards, what is the probability the hand contains exactly one heart? [0.4114]
- 1.54.** If a five card hand is dealt from a deck of 52 cards, what is the probability the hand contains at least one heart? [0.77847]
- 1.55.** Seven people are randomly selected from a group of 10 men and 11 women to form a committee. What is the probability that exactly 5 males are on the committee? [0.11919]
- 1.56.** Five balls are drawn without replacement from a bag containing 3 metal balls and 5 glass balls. What is the probability that at least 3 glass balls are drawn? [0.82143]
- 1.57.** A bookcase contains 6 different maths books and 12 different physics books. If a student randomly selects two of these books, what is the probability that they are both maths books or both physics books? [0.52941]

Chapter 2

Probability Theory

2.1 Probability

Definition 2.1 (Axiomatic definition of probability). Let γ be a space of elementary events and let τ be a sample space over γ . Let the following axioms be fulfilled:

A₁: to each event $A \in \tau$ is assigned exactly one non-negative number $P(A)$ which is called a *probability of event A*;

A₂: $P(I) = 1$;

A₃: for each system of disjoint events $A_1, A_2, \dots, A_n, \dots \in \tau$ holds

$$P(A_1 \cup A_2 \cup \dots \cup A_n \cup \dots) = P(A_1) + P(A_2) + \dots + P(A_n) + \dots \quad (2.1)$$

We call the triplet $[\gamma, \tau, P]$ a *probability space*.

These axioms do not specify an exact prescription to determine the probability of a phenomenon. In specific situations, we use the classical definition of probability.

It is easy to see that the classical definition of the probability is consistent with the axioms **A1** – **A3**. We can prove the assertions of the following theorem using axioms **A1** – **A3**.

Theorem 2.1. Let $[\gamma, \tau, P]$ be a probability space. The following assertions hold:

1. $P(\emptyset) = 0$;
2. $P(\overline{A}) = 1 - P(A)$ for each $A \in \tau$;
3. if $A \subseteq B$ then $P(A) \leq P(B)$;
4. $0 \leq P(A) \leq 1$ for each $A \in \tau$;
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for each $A, B \in \tau$;

6. for each $A, B, C \in \tau$ the following equality holds:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C);$$

7. the generalization: for each $A_1, A_2, \dots, A_n \in \tau$

$$\begin{aligned} P\left(\bigcup_{i=1}^n A_i\right) &= \sum_{i=1}^n P(A_i) - \sum_{i,j=1, i < j}^n P(A_i \cap A_j) + \sum_{i,j,k=1, i < j < k}^n P(A_i \cap A_j \cap A_k) - \dots \\ &\quad \dots + (-1)^{n+1} P\left(\bigcap_{i=1}^n A_i\right). \end{aligned}$$

Proof. For the demonstration example we prove only the first claim. From the properties of sample space it follows that $\emptyset \in \tau$ and from \mathbf{A}_1 we get the existence of $P(\emptyset)$. Since $I \in \tau$, then $I \cup \emptyset \in \tau$ and from \mathbf{A}_1 there exists $P(I \cup \emptyset)$. Events I, \emptyset are disjoint, so $P(I \cup \emptyset) = P(I) + P(\emptyset)$ according to \mathbf{A}_3 . Obviously, $I \cup \emptyset = I$, which implies $P(I) = P(I) + P(\emptyset)$ and consequently $P(\emptyset) = 0$. \square

2.2 Conditional Probability

The conditional probability of an event A given B is the probability that the event A will occur given the knowledge that the event B has already occurred. This probability is denoted by $P(A|B)$, notation for the probability of A given B . In the case where events A and B are independent (where event B has no effect on the probability of event A), the conditional probability of event A given B is equal to the probability of event A , that is $P(A)$.

Definition 2.2 (Conditional probability). Let $[\gamma, \tau, P]$ be a probability space. The *probability of A given B* is defined as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \quad (2.2)$$

If $P(B) = 0$ then $P(A|B)$ is not defined.

If $P(A) \neq 0$ and $P(B) \neq 0$ then $P(B|A) = \frac{P(A \cap B)}{P(A)}$. Using this equality together with (2.2) we get $P(A \cap B) = P(A) \cdot P(B|A)$ and $P(A \cap B) = P(B) \cdot P(A|B)$ which implies

$$P(A|B) \cdot P(B) = P(B|A) \cdot P(A). \quad (2.3)$$

The following theorem is a generalization of the previous reasoning.

Theorem 2.2 (Probability of intersection). Let $[\gamma, \tau, P]$ be a probability space and let $A_1, A_2, \dots, A_n \in \tau$. Then

$$P\left(\bigcap_{i=1}^n A_i\right) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdot \dots \cdot P(A_n|A_1 \cap A_2 \cap \dots \cap A_{n-1}) \quad (2.4)$$

The relationship between the hypothesis and the conditional probability is expressed by the following theorem.

Theorem 2.3 (Total Probability Law). Let $[\gamma, \tau, P]$ be a probability space and let H_1, H_2, \dots, H_n be a complete system of hypothesis. Then for each $A \in \tau$ the following equality holds:

$$\begin{aligned} P(A) &= P(H_1) \cdot P(A|H_1) + P(H_2) \cdot P(A|H_2) + \dots + P(H_n) \cdot P(A|H_n) = \\ &= \sum_{i=1}^n P(H_i) \cdot P(A|H_i). \end{aligned} \quad (2.5)$$

Proof. For an arbitrary event A we have

$$A = A \cap I = A \cap (H_1 \cup H_2 \cup \dots \cup H_n) = (A \cap H_1) \cup (A \cap H_2) \cup \dots \cup (A \cap H_n).$$

Since events H_i are mutually disjoint and $A \cap H_i \subset H_i$, events $(A \cap H_1), (A \cap H_2), \dots, (A \cap H_n)$ are mutually disjoint, too. According to A_3 we get

$$P(A) = P(A \cap H_1) + P(A \cap H_2) + \dots + P(A \cap H_n).$$

We obtain Equality (2.5) by substituting $P(A \cap H_i) = P(H_i) \cdot P(A|H_i)$ for all $i = 1, 2, \dots, n$.

Theorem 2.4 (Bayes' Theorem). Let $[\gamma, \tau, P]$ be a probability space, H_1, H_2, \dots, H_n be hypothesis and $A \in \tau$ be such that $P(A) \neq 0$. Then for each $H_k, k = 1, 2, \dots, n$ the following equality holds:

$$P(H_k|A) = \frac{P(H_k) \cdot P(A|H_k)}{\sum_{i=1}^n P(H_i) \cdot P(A|H_i)} = \frac{P(H_k) \cdot P(A|H_k)}{P(A)}. \quad (2.6)$$

2.3 Independent Events

2.3.1 Probability of Independent Events

Definition 2.3 (Independent events). Two events A, B are *independent* if the occurrence of one of them does not affect the probability of the other or if the probability of one of them equals zero, i.e., one of the following possibilities occurs:

$$P(A|B) = P(A) \quad \text{or} \quad P(B) = 0 \quad \text{or} \quad P(B|A) = P(B) \quad \text{or} \quad P(A) = 0. \quad (2.7)$$

Two events are *dependent* if the outcome or occurrence of the first affects the outcome or occurrence of the second so that the probability is changed.

Theorem 2.5. Let $[\gamma, \tau, P]$ be a probability space. Events $A, B \in \tau$ are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B). \quad (2.8)$$

Remark 2.1.

1. Equality (2.8) gives an equivalent condition for the independence of events, and it is frequently used as a definition of independent events.
2. The independence of events A and B implies the independence of these pairs of events:

$$A \text{ and } \bar{B}, \quad \bar{A} \text{ and } B, \quad \bar{A} \text{ and } \bar{B}.$$

Definition 2.4. The system of events A_1, A_2, \dots, A_n is *overall independent*, if the probability of occurrence of one of them does not change upon the occurrence of any group of other events or if the probability of one of them equals zero.

Theorem 2.6 (Probability of intersection of overall independent events). Let $[\gamma, \tau, P]$ be a probability space. The system of events A_1, A_2, \dots, A_n is overall independent, if one of the following conditions is satisfied:

1. for each $k \leq n$ holds

$$P\left(\bigcap_{j=1}^k A_{i_j}\right) = \prod_{j=1}^k P(A_{i_j}), \quad (2.9)$$

- 2.

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \dots \cdot P(A_n). \quad (2.10)$$

Theorem 2.7 (Probability of union of overall independent events). Let $[\gamma, \tau, P]$ be a probability space. If the system of events A_1, A_2, \dots, A_n is overall independent then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = 1 - P(\bar{A}_1) \cdot P(\bar{A}_2) \cdot \dots \cdot P(\bar{A}_n). \quad (2.11)$$

2.3.2 Probability of Repeated Independent Trials

Many experiments share the common element that their outcomes can be classified into one of two events, e.g., a coin can come up head or tail; a child can be male or female; a person can be employed or unemployed. These outcomes are often labeled as “success” or “failure”. Note that there is no connotation of “goodness” here — for example, when looking at births, the statistician might label the birth of a boy as a “success” and the birth of a girl as a “failure” but the parents would not necessarily see things that way. The usual notation is p is probability of success, q is probability of failure. Note that $p + q = 1$. We are often interested in the result of independent, repeated trials, i.e., the number of successes in repeated trials.

For example, we can determine the probability of getting 2 numbers “6” in 5 die rollings. At first, we have to determine the probability of one possible way the event can occur, and then determine the number of different ways the event can occur. That is,

$$P(\text{event}) = (\text{number of ways event can occur}) \cdot P(\text{one occurrence}).$$

In our example we will call getting a “6” a “success.” Also, in this case, the number of repetitions is $n = 5$, the number of successes is $r = 2$, and the number of failures is $n - r = 5 - 2 = 3$. One way this can occur is if the first 2 rollings are “6” and the last three are not “6”. The probability is $(1/6)^2 \cdot (5/6)^3$. The number of ways event can occur is $\binom{5}{2}$, so the probability of this event is

$$P = \binom{5}{2} \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^3 = \frac{3750}{6^5} = 0.48335.$$

Theorem 2.8 (Bernoulli theorem). Let p be the probability that the outcome of trial is "success" considering event A and $P_{n,p}(k)$ be the probability that repeating a trial n times the number of "success" outcomes is k . Then

$$P_{n,p}(k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad \text{for } k = 0, 1, \dots, n. \quad (2.12)$$

2.4 Solved Examples

Example 2.1. In a math class of 30 students, 17 are boys and 13 are girls. On a unit test, 4 boys and 5 girls made an “A” grade. If a student is chosen at random from the class, what is the probability of choosing a girl or an “A” student?

Solution:

Let A be the event that chosen student made “A” and B be the event that chosen student is a girl. We want to compute $P(B \cup A)$. We get

$$P(B \cup A) = P(B) + P(A) - P(B \cap A) = \frac{13}{30} + \frac{9}{30} - \frac{5}{30} = \frac{17}{30}.$$

✓

Example 2.2. Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of the defective fuses in the first two tests?

Solution:

Let A be the event that we find a defective fuse in the first test and B be the event that we find a defective fuse in the second test. We are told that $P(A) = \frac{2}{7}$ and $P(B|A) = 1/6$. We want to compute $P(A \cap B)$. We get

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{21} = 0.047619.$$

✓

Example 2.3. Mr. Parietti needs three students to help him with a science demonstration for his class of 18 girls and 12 boys. He randomly chooses one student who comes to the front of the room. Then he chooses a second student from those still seated and finally he chooses a third student from those still seated. What is the probability that all three students chosen are girls?

Solution:

Let A_i be the event that i th chosen student is a girl. We compute

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) = \frac{18}{30} \cdot \frac{17}{29} \cdot \frac{16}{28} = \frac{4896}{24360} = 0.20099.$$

✓

Example 2.4. If six cards are selected at random (without replacement) from a standard deck of 52 cards, what is the probability that there will be no pairs (two cards of the same denomination)?

Solution:

Let A_i be the event that the first i cards have no pair among them. Then we want to compute $P(A_6)$. We have $P(A_1) = 1$ and $A_{i+1} \subset A_i$ which implies $A_1 \cap A_2 = A_2$, $A_1 \cap A_2 \cap A_3 = A_3, \dots$. In general, $A_1 \cap A_2 \cap \dots \cap A_n = A_n$. We have $P(A_2) = P(A_1) \cdot P(A_2|A_1) = 48/51$ because there are 51 cards and 48 of them has another denomination than the first card. At the same manner we get

$$P(A_3) = P(A_2 \cap A_3) = P(A_2) \cdot P(A_3|A_2) = \frac{48}{51} \cdot \frac{44}{50}, \dots,$$

$$P(A_6) = P(A_2) \cdot P(A_3|A_2) \cdot P(A_4|A_3) \cdot P(A_5|A_4) \cdot P(A_6|A_5) = \frac{48}{51} \cdot \frac{44}{50} \cdot \frac{40}{49} \cdot \frac{36}{48} \cdot \frac{32}{47} = 0.34524.$$

But, there is also another possibility for determining the possibility using the classical definition of probability only. Since the selection is without replacement, the number of a elementary events $n = \binom{52}{6}$. The number of elementary events of which consists event A is $m = \binom{13}{6} \cdot 4^6$, since there are $\binom{13}{6}$ possibilities for choosing 6 different denominations and for each of them there are 4 cards. We get

$$P(A) = \frac{\binom{13}{6} \cdot 4^6}{\binom{52}{6}} = 0.34524.$$

✓

Example 2.5. Two cards from an ordinary deck of 52 cards are missing. What is the probability that a random card drawn from this 50-card deck is a spade?

Solution:

Let A be the event that the randomly drawn card is a spade. Let H_i be the event that i spades are missing from the 50-card (defective) deck for $i = 0, 1, 2$. We want to compute

$P(A)$, which we compute by conditioning on how many spades are missing from the original (good) deck and by using the total law probability:

$$\begin{aligned} P(A) &= P(A|H_0) \cdot P(H_0) + P(A|H_1) \cdot P(H_1) + P(A|H_2) \cdot P(H_2) = \\ &= \frac{13}{50} \cdot \frac{\binom{13}{0} \cdot \binom{39}{2}}{\binom{52}{2}} + \frac{12}{50} \cdot \frac{\binom{13}{1} \cdot \binom{39}{1}}{\binom{52}{2}} + \frac{11}{50} \cdot \frac{\binom{13}{2} \cdot \binom{39}{0}}{\binom{52}{2}} = 1/4. \end{aligned}$$

✓

Example 2.6. One half percent of the population has a particular disease. A test is developed for the disease. The test gives a false positive 3% of the time and a false negative 2% of the time.

- What is the probability that Joe (a random person) tests positive?
- If Joe's test came back positive, what is the probability that Joe has the disease?

Solution:

Let H_1 be the event that Joe has the disease and H_2 be the event that Joe has not the disease. Let A be the event that Joe's test comes back positive. We are told that $P(H_1) = 0.005$ since 0.5% of the population has the disease. This follows that $P(H_2) = 0.995$. We are also told that $P(A|H_1) = 0.98$, since 2% of persons having the disease is missed ("false negative"). We are told that $P(A|H_2) = 0.03$, since there are 3% "false positives".

- We want to compute $P(A)$. According to the total law probability we compute

$$P(A) = P(A|H_1) \cdot P(H_1) + P(A|H_2) \cdot P(H_2) = 0.98 \cdot 0.005 + 0.03 \cdot 0.995 = 0.03475.$$

- We have to compute $P(H_1|A)$. According to the Bayes' theorem we get

$$P(H_1|A) = \frac{P(A|H_1) \cdot P(H_1)}{P(A|H_1) \cdot P(H_1) + P(A|H_2) \cdot P(H_2)} = \frac{0.98 \cdot 0.005}{0.03475} = 0.13818.$$

✓

Example 2.7. Urn 1 contains 5 white balls and 7 black balls. Urn 2 contains 3 white and 12 black balls. A fair coin is flipped; if it is head, a ball is drawn from Urn 1, and if it is tail, a ball is drawn from Urn 2. Suppose that this experiment is done and you learn that a white ball was selected. What is the probability that this ball was taken from Urn 2?

Solution:

Let H_1 be the event that the coin flip was tails and H_2 be the event that the coin flip was heads. Let A be the event that a white ball is selected. From the given data, we know that $P(A|H_1) = 5/12$ and that $P(A|H_2) = 3/15 = 1/5$. Since the coin is fair, we know

that $P(H_1) = P(H_2) = 1/2$. We want to compute $P(H_2|A)$, which we do using the Bayes' formula:

$$P(H_2|A) = \frac{(1/5) \cdot (1/2)}{(1/5) \cdot (1/2) + (5/12) \cdot (1/2)} = \frac{12}{37} = 0.3243.$$

✓

Example 2.8. Charlie has 2 children. He picks one of them at random and comes by your house; he brings his son Joe. What is the probability that Joe's sibling is a brother?

Solution:

We are given the event A that "Charlie randomly choose one of his 2 children, and that chosen child is a son". Let H_1 be the event that Charlie has two boys, H_2 be the event that Charlie has two girls, and H_3 be the event that Charlie has one girl and one boy. Supposing that the birth of a boy is just as likely as the birth of a girl we have $P(H_1) = P(H_2) = 1/4$, $P(H_3) = 1/2$. The conditional probabilities are $P(A|H_1) = 1$, $P(A|H_2) = 0$ and $P(A|H_3) = 1/2$. We compute

$$\begin{aligned} P(H_1|A) &= \frac{P(H_1 \cap A)}{P(A)} = \frac{P(H_1)}{\sum_{i=1}^3 P(A|H_i) \cdot P(H_i)} = \\ &= \frac{1/4}{1 \cdot (1/4) + 0 \cdot (1/4) + (1/2) \cdot (1/2)} = 1/2. \end{aligned}$$

✓

Example 2.9. Consider the game of Let's make a deal in which there are three doors (numbered 1, 2, 3), one of which has a car behind it and two of which are empty (have "booby prizes"). You initially select Door 1, then, before it is opened, Monty Hall tells you that Door 3 is empty (has a booby prize). You are then given the option to change your selection from Door 1 to the unopened Door 2. What is the probability that you will win the car if you change your door selection to Door 2?

Solution:

Let A be the event that the car is behind Door 2 and B be the event that Door 3 is empty. The probability that you win by changing to Door 2, given that he tells you Door 3 is empty is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{2/3} = 1/2$$

✓

Example 2.10. Consider the game of Let's Make a Deal in which there are five doors (numbered 1, 2, 3, 4, and 5), one has a car behind it and four are empty (have "booby prizes"). You initially select Door 1, then, before it is opened, Monty Hall opens Door 2 and Door 4 that are empty (selecting the two at random if there are three empty doors among 2, 3, 4, 5). (We are assuming that Monty Hall knows where the car is and that he

selects doors to open only from among those that are empty.) You are then given the option to change your selection from Door 1 to Door 3. Given that Monty opens Door 2 and Door 4, what is the probability that you will win the car if you change your door selection to Door 3? Also, compute the probability that you will win a car if you do not change. What is the better, change to Door 3 or stay with Door 1?

Solution:

Let H_i be the event that you will win a car by selecting Door i . Let A be the event that Monty shows you Door 2 and Door 4. The probability that you win by changing to Door 3, given that he shows you Door 2 and Door 4 is:

$$\begin{aligned} P(H_3|A) &= \frac{P(A|H_3) \cdot P(H_3)}{\sum_{i=1}^5 P(A|H_i) \cdot P(H_i)} = \\ &= \frac{1/\binom{3}{2} \cdot 1/5}{1/\binom{4}{2} \cdot 1/5 + 0 + 1/\binom{3}{2} \cdot 1/5 + 0 + 1/\binom{3}{2} \cdot 1/5} = \frac{2}{5}. \end{aligned}$$

The probability that you win staying with Door 1, given that he shows you Door 2 and Door 4 is:

$$\begin{aligned} P(H_1|A) &= \frac{P(A|H_1) \cdot P(H_1)}{\sum_{i=1}^5 P(A|H_i) \cdot P(H_i)} = \\ &= \frac{1/\binom{4}{2} \cdot 1/5}{1/\binom{4}{2} \cdot 1/5 + 0 + 1/\binom{3}{2} \cdot 1/5 + 0 + 1/\binom{3}{2} \cdot 1/5} = \frac{1}{5}. \end{aligned}$$

Since $P(H_3|A) > P(H_1|A)$, it is better to change the selection from Door 1 to Door 3. \checkmark

Example 2.11. Two machines produce the same products, with the productivity of the first being three times higher than the productivity of the second. The first machine produces 70% of quality products, the first one 80% of quality products. Find the probability that

- a randomly selected product is good;
- a randomly selected product is produced by the second machine, if it is good.

Solution:

Let H_i be the event that the product is made by i th machine and let A denotes the event that a product is good. Since the first machine produces 3/4 of products and the second machine produces 1/4 of products, we have $P(H_1) = 0.75$ and $P(H_2) = 0.25$. The conditional probabilities are $P(A|H_1) = 0.7$ and $P(A|H_2) = 0.8$.

- We compute $P(A)$ by total probability law:

$$P(A) = P(A|H_1) \cdot P(H_1) + P(A|H_2) \cdot P(H_2) = 0.7 \cdot 0.75 + 0.8 \cdot 0.25 = 0.725.$$

b) Using Bayes' formula we obtain

$$P(H_2|A) = \frac{P(A|H_2) \cdot P(H_2)}{P(A)} = \frac{0.8 \cdot 0.25}{0.725} = 0.2759.$$

✓

Example 2.12. Using the formal definition of independence, determine whether events A and B are independent or dependent.

Rolling two dice, with

Event A: Rolling 1 on the first die.

Event B: The dice summing to 7.

Solution:

We have

$$P(A) = 1/6, \quad P(B|A) = 1/6, \quad P(A \cap B) = \frac{1}{36}.$$

Since $P(A) \cdot P(B|A) = P(A \cap B)$, the events are independent.

✓

Example 2.13. Determine whether events A and B are independent or dependent.

Flip three coins, with

Event A: The first two coins are heads.

Event B: There are at least two heads among the three coins.

Solution:

We have

$$P(A) = \frac{2}{8} = \frac{1}{4}, \quad P(B) = \frac{4}{8} = \frac{1}{2}, \quad P(A \cap B) = \frac{2}{8} = \frac{1}{4}.$$

Since $P(A) \cdot P(B) \neq P(A \cap B)$, according to Theorem 2.5 the events are dependent. ✓

Example 2.14. Four basketball players throw on the basket. The probability of successful hit for individual basketball players is 0.8; 0.7; 0.85 and 0.9. Calculate the probability that

- all four hit the basket;
- no one hits the basket;
- at least one hits the basket;
- at least one does not hit the basket.

Solution:

Denote by A_i the event that i -th basketball player hits the basket. We are told that

$$P(A_1) = 0.8, \quad P(A_2) = 0.7, \quad P(A_3) = 0.85, \quad P(A_4) = 0.9.$$

Teda

$$\text{a) } P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) = 0.8 \cdot 0.7 \cdot 0.85 \cdot 0.9 = 0.4284$$

$$\text{b) } P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4) = P(\bar{A}_1) \cdot P(\bar{A}_2) \cdot P(\bar{A}_3) \cdot P(\bar{A}_4) = 0.2 \cdot 0.3 \cdot 0.15 \cdot 0.1 = 0.0009$$

$$\text{c) } P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - P(\bar{A}_1) \cdot P(\bar{A}_2) \cdot P(\bar{A}_3) \cdot P(\bar{A}_4) = 0.9991$$

$$\text{d) } P(\bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3 \cup \bar{A}_4) = 1 - P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) = 0.5716$$

✓

Example 2.15. Suppose you flip a fair coin 6 times. What is the probability that the coin will land head exactly 4 times?

Solution:

We have the number of trials $n = 6$, the number of successes $k = 4$ and the probability $p = 1/2$. According to (2.12) we have

$$P_6(4) = \binom{6}{4} \left(\frac{1}{2}\right)^4 \cdot \left(1 - \frac{1}{2}\right)^{6-4} = 0.234375.$$

✓

Example 2.16. Suppose you toss a pair of dice 8 times. What is the probability that the dice will get a sum of 7 at most 3 times?

Solution:

The number of ways you can have a sum 7 is 6, since the two dice can land as (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1). There are 36 ways the two dice can land, so the probability of success is $p = 1/6$. We have $n = 8$, $k = 0, 1, 2, 3$ and by (2.12) we obtain

$$\begin{aligned} P(A) = P_8(0) + P_8(1) + P_8(2) + P_8(3) &= \binom{8}{0} \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^8 + \binom{8}{1} \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^7 + \\ &+ \binom{8}{2} \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^6 + \binom{8}{3} \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^5 = 0.96934. \end{aligned}$$

✓

Example 2.17. An apartment building has residents living on the second and third floors. The residents use an elevator to get to their apartments. There are 20 people living on the second floor. The third floor has more luxurious, larger apartments and there are 6 people living on the third floor. Six random residents use the elevator to get to their apartments. What is the probability that exactly 4 of the people exit on the second floor and that 2 people exit on the third floor?

Solution:

We consider each time one of the resident uses the elevator as a separate event. We will

call an exiting on the second floor a “success.” So the probability of success is $p = 20/26 = 10/13$. We have $n = 6$, $k = 4$ and by (2.12) we obtain

$$P_6(4) = \binom{6}{4} \left(\frac{10}{13}\right)^4 \cdot \left(\frac{3}{13}\right)^2 = 0.194.$$

✓

2.5 Unsolved tasks

2.1. You decide to tell your fortune by drawing two cards from a standard deck of 52 cards. What is the probability of drawing two cards of the same suite in a row? The cards are not replaced in the deck. [0.23529]

2.2. What is the probability of drawing two aces from a standard deck of cards, given that the first card is an ace? The cards are not returned to the deck. [0.05882]

2.3. A new superman MasterCard has been issued to 2000 customers. Of these customers, 1500 hold a Visa card, 500 hold an American Express card and 40 hold a Visa card and an American Express card. Find the probability that a customer chosen at random holds a Visa card, given that the customer holds an American Express card. [0.00118]

2.4. The Census Bureau has estimated 80% of men lives at least 70 years and 50% lives at least 80 years. What is the probability that a man lives at least 80 years given that he has just celebrated his 70th birthday? [0.629]

2.5. There are two urns containing coloured balls. The first urn contains 50 red balls and 50 blue balls. The second urn contains 30 red balls and 70 blue balls. One of the two urns is randomly chosen and then a ball is drawn at random from one of the two urns.

a) What is the probability, that a red ball is drawn? [0.4]

b) If a red ball is drawn, what is the probability that it comes from the first urn? [0.625]

2.6. An economics consulting firm has created a model to predict recessions. The model predicts a recession with probability 80% when a recession is indeed coming and with probability 10% when no recession is coming. The probability of falling into a recession is 20%. If the model predicts a recession, what is the probability that a recession will indeed come? [0.6]

2.7. Alice has two coins in her pocket, a fair coin (head on one side and tail on the other side) and an unfair two-headed coin. She picks one at random from her pocket, flips it and obtains head. What is the probability that she flipped the fair coin? [0.3]

2.8. A firm undertakes two projects, A and B . The probabilities of having a successful outcome are $3/4$ for project A and $1/2$ for project B . The probability that both projects will have a successful outcome is $7/16$. Are the two outcomes independent?

[not independent]

2.9. A firm undertakes two projects, A and B . The probabilities of having a successful outcome are $2/3$ for project A and $4/5$ for project B . What is the probability that neither of the two projects will have a successful outcome if their outcomes are independent? [0.06]

2.10. A card is chosen at random from a standard deck of 52 playing cards. Without replacing it, a second card is chosen. What is the probability that the first card chosen is a queen and the second card chosen is a jack? [0.006]

2.11. Four cards are chosen at random from a deck of 52 cards without replacement. What is the probability of choosing a ten, a nine, an eight and a seven in this order? [0.000394]

2.12. A survey determines that in a particular town, 33% of the residents jog, 42% bike, and 12% do both activities. What is the probability that a randomly selected person does neither activity? [0.37]

2.13. A basket contains 20 fruits of which 10 are oranges, 8 are apples, and 2 are tangerines. You randomly select 5 fruits and give them to your friend. What is the probability that among the 5 fruits, your friend will get 2 tangerines? [0.05263]

2.14. A school survey found that 7 out of 30 students walk to school. If four students are selected at random without replacement, what is the probability that the first and the second walk to school, but the third and the fourth do not walk to school? [0.03231]

2.15. The probability of a New York teenager owning a skateboard is 0.37, of owning a bicycle is 0.81 and of owning both is 0.36. If a New York teenager is chosen at random, what is the probability that the teenager owns a skateboard or a bicycle? [0.82]

2.16. A single 6-sided die is rolled. What is the probability of rolling a number greater than 3 or an even number? [0.6]

2.17. In a junior football league, 55% of the players come from Western Canada, and 45% are from Eastern Canada. From this league, 17% of the Western players and 11% of the Eastern players will go to the CFL.

a) What is the probability that a randomly chosen player will go to CFL? [0.143]

b) If a randomly chosen CFL player who came from the junior league is selected, what is the probability that he came from Eastern Canada? [0.34615]

2.18. The probability of having a particular disease is 5%. The test to determine if a person has this disease is 83% accurate. What is the probability that a randomly selected person tests positive? [0.203]

2.19. If 3% of the population has a specific disease, and the test for this disease is 92% accurate, what is the probability a person does not have the disease given that the test result is positive? [0.73764]

2.20. Bag A contains three metal balls and six glass balls, and bag B contains four metal balls and three glass balls. In a game, a person rolls a die to determine which bag to pull a ball out of. If the die rolls a 1, 2 or 3, the ball is pulled from bag A. If the die comes up 4, 5, or 6, the ball is pulled from bag B.

a) What is the probability that a glass ball is selected? [0.54762]

b) If the ball selected is made of glass, what is the probability it came from bag B?

[0.3913]

2.21. Bag A contains four metal balls and six glass balls, and bag B contains five metal balls and two glass balls.

a) A ball is randomly selected from bag A and placed in bag B. A ball is then pulled at random out of bag B. What is the probability that the ball from bag B is metal?

[0.675]

b) If a metal ball was selected from bag B, what is the probability that a glass ball was transferred from bag A to bag B? [0.5]

2.22. A grocery store obtains 35% of its products from vendor A, and 65% of its produce from vendor B. It is expected that spoilage will result in 12% of vendor A's produce and 17% of vendor B's products to be discarded. What is the probability a randomly picked product came from vendor A, given that it was picked from the discard pile? [0.27541]

2.23. A security code consists of 8 digits, which may be any number from 0 to 9. The first digit is allowed to be zero and repetitions are allowed. What is the probability a particular code begins with exactly two 7's, to the nearest hundredth? [0.01]

2.24. Captain Tiffany has a ship. The ship is two furlongs from the dread pirate Umaima and her merciless band of thieves. The Captain has probability $\frac{3}{5}$ of hitting the pirate ship. The pirate only has one good eye, so she hits the Captain's ship with probability $\frac{4}{9}$. If both fire their cannons at the same time, what is the probability that the Captain hits the pirate ship, but the pirate misses? [0.3]

2.25. Carmelo Anthony is shooting free throws. Making or missing free throws doesn't change the probability that he will make his next one, and he makes his free throws 84% of the time. What is the probability of Carmelo Anthony making none of his next 5 free throw attempts? [0.000104]

2.26. Carlos Boozer is shooting free throws. Making or missing free throws doesn't change the probability that he will make his next one, and he makes his free throws 70% of the time. What is the probability of Carlos Boozer making 2 of his next 5 free throw attempts? [0.1323]

2.27. A test consists of 10 multiple choice questions with five choices for each question and exactly one correct answer for each question. As an experiment, you GUESS on each and every answer without even reading the questions. What is the probability of getting exactly 6 questions correct on this test? [0.0005505]

2.28. At a certain intersection, the light for eastbound traffic is red for 15 seconds, yellow for 5 seconds, and green for 30 seconds. Find the probability that out of the next eight eastbound cars that arrive randomly at the light, exactly three will be stopped by a red light. [0.25412]

2.29. When rolling a die 100 times, what is the probability of rolling a "4" exactly 25 times? [0.009829]

2.30. A doctor notes that 20% of the patients he tests actually have mononucleosis. When a patient has mononucleosis, the test shows a positive result (indicating disease presence) 90% of the time. When a patient does not have the disease, the test shows a positive result 10% of the time. If a patient's test result is positive, what is the probability that the patient actually has the mononucleosis? [0.69231]

2.31. Four radar systems are arranged so that they work independently of each other. Each system has a 0.9 chance of detecting an approaching airborne object. Find the probability that at least one radar system will fail to detect an approaching object. [0.3439]

2.32. A federal agency is trying to decide which of two waste dump projects to investigate. An administrator estimates that the probability of federal law violations in the first project is 0.3. She also estimates that the probability of violations in the second project is 0.25. In addition, she believes the occurrence of violations in these two projects are mutually exclusive. What is the probability of federal law violations in the first project or in the second project or both? [0.475]

2.33. A manufacturer of hand soap has introduced a new product. An extensive survey indicates that 40% of the people have seen advertising for the new product. It also showed that 20% of the people in the survey had tried the new product. In addition, 15% of those in the survey had seen it advertised and had tried the product. What is the probability that a randomly chosen person would have seen the advertising for the new product or have tried the product or both? [0.45]

2.34. Box I contains 7 red and 3 white balls and box II contains 2 red and 6 white balls. First a box is selected at random (each box is as likely to be selected as the other) and then a ball is drawn from the box. If a red ball is drawn, what is the probability that it came from box I? [0.73684]

2.35. Probability can be used to interpret the results of a test given to determine if a person is using drugs. Suppose we assume that 5% of people are drug users. A test is 96% accurate, which we'll say means that if a person is a drug user, the result is positive 96% of the time, and if the person is not a drug user, the result is negative 96% of the time. A randomly chosen person tests positive. What is the probability that the person is a drug user? [0.55814]

2.36. Suppose that the test for drug use is given to a group of people who are known to have a higher likelihood of being drug users. Assume 40% of the people in this group are drug users. A test is 98% accurate. What is the probability that an individual in this group who tests positive really is not a drug user? [0.0297]

2.37. A manufacturer obtains computer modems from three different subcontractors: 30% from A, 45% from B, and 25% from C. Past experience has shown that the defective rates for these subcontractors are 3%, 1%, and 4% respectively. If a modem is returned to the manufacturer by a customer because it was found to be defective, what is the probability that it came from subcontractor C? [0.42553]

2.38. A pet store owner sells specially clothes for pets. From past data 5% of customers buy specially clothes for their pets. What is the probability that at least 4 of the first 20 customers buy specially clothes for their pets? [0.0159]

2.39. The owner of a small convenience store notices that only 5% of customers buy magazines.

a) What is the probability that the first customer to buy a magazine is the 4th customer? [0.04287]

b) What is the probability that the first customer to buy a magazine is the 8th customer? [0.03492]

2.40. In the typical bag of candies are 6 brown candies, 4 red candies, 4 yellow candies, 2 green candies, 2 orange candies and 2 blue candies. You have just purchased a bag of candies and select one candy at a time from the bag. What is the probability that

- a) the first red one is the 4th candy you select from the bag, [0.11558]
- a) the first blue one is the 3th candy you select from the bag? [0.08947]

2.41. A student takes a multiple choice test with 20 questions, each with 5 choices (only one of which is correct). Suppose that the student blindly guesses. What is the probability that the student answers

- a) exactly 10 questions correctly, [0.00203]
- b) at most 5 questions correctly, [0.80421]
- c) at least 5 questions correctly? [0.37035]

2.42. Suppose you independently flip a coin 4 times and the outcome of each toss can be either head or tails. What is the probability of obtaining exactly 2 tails? [0.375]

2.43. Suppose you independently throw a dart 10 times. Each time you throw a dart, the probability of hitting the target is $\frac{3}{4}$. What is the probability of hitting the target

- a) exactly 5 times, [0.05839]
- b) less than 5 times, [0.01973]
- c) more than 5 times? [0.92188]

2.44. A standard, fair die was tossed 10 times. What is the probability that

- a) the number 6 was showed at least three times, [0.22478]
- b) the number greater than 4 was showed exactly five times, [0.13656]
- c) the odd number was showed at most four times? [0.37695]

2.45. The probability for a child to catch a certain disease is 20%. Find the probability that the 12th child exposed to the disease will be the 3rd to catch it. [0.05906]

2.46. Suppose that 20 patients arrive at a hospital on any given day. Assume that 10% of all the patients of this hospital are emergency cases. Find the probability that

- a) exactly 5 of the 20 patients are emergency cases, [0.03192]
- b) none of the 20 patients is emergency case, [0.12157]
- c) all 20 patients are emergency cases, [10^{-20}]
- d) at least 4 of the 20 patients are emergency cases, [0.13295]

e) more than 4 of the 20 patients are emergency cases. [0.04317]

2.47. Patients arrive at a hospital. Assume that 10% of all the patients of this hospital are emergency cases.

a) Calculate the probability that at any given day the 20th patient will be the first emergency case. [0.01351]

b) Calculate the probability that the first emergency case will occur after the arrival of the 20th patient. [0.12157]

c) Find the probability that the first emergency case will occur on or before the arrival of the 15th patient. [0.79411]

2.48. An apartment building has 11 residents. There are 7 residents on the second floor and 4 residents on the third floor. Six people use the elevator at random to go to their apartments. What is the probability that at most 3 of the residents exit on the second floor? [0.35404]

2.49. A die is rolled 8 times. Given that there were 3 sixes in the 8 rolls, what is the probability that there were 2 sixes in the first 5 rolls? [0.53571]

2.50. A man fires 8 shots at a target. Assume that the shots are independent, and each shot hits the bull's eye with probability 0.7.

a) What is the chance that he hits the bull's eye exactly 4 times? [0.13614]

b) Given that he hits the bull's eye at least twice, what is the chance that he hits the bull's eye exactly 4 times? [0.13632]

c) Given that the first two shots hit the bull's eye, what is the chance that he hits the bull's eye exactly 4 times in 8 shots? [0.05959]

2.51. A gambler decides to keep betting on red at roulette, and stop as soon as she has won a total five bets.

a) What is the probability that she has to make exactly 8 bets before stopping? [0.13672]

b) What is the probability that she has to make at least 9 bets? [0.63672]

2.52. Let 4: 6: 5: 5 be the ratio of the number of products produced on four machines. The probabilities of producing a first-quality product for individual machines are 0.7; 0.6; 0.8 and 0.7, respectively.

a) What is the probability that a randomly selected product is of first quality? [0.695]

b) Calculate the probability that a randomly selected product was made on the first machine, is it of first quality. [0.2014]

2.53. For math school-leaving exam, students have 30 questions. The student learned 12 questions on average and 18 questions master well. The student answers two questions. The probability that the student will get at least a B rating is

- 0.3 if two average learned questions are asked;
- 0.8 if two master well learned questions are asked;
- 0.5 if one master well and one average learned questions are asked.

- a) What is the probability that a student will get at least a B rating? [0,5752]
- b) What is the probability that two average learned questions were asked, if a student received at least a B rating? [0,0791]

2.54. There are 60 percent of girls in the classroom. Of these, 70% is less than 170 cm. Of the boys, only 30% are lower than 170 cm.

- a) Calculate the probability that a randomly selected pupil is less than 170 cm. [0,54]
- b) Calculate the probability that a randomly selected pupil is a boy if he is less than 170 cm. [0,222]

2.55. The week has 5 working days and 2 free days (weekend). The probability that a student gets up before 6³⁰ during workday is 0.7. The probability that a student gets up before 6³⁰ during the weekend is 0.1.

- a) Calculate the probability that a student gets up before 6³⁰. [0,5285]
- b) What is the probability that the working day was when a student got up before 6³⁰? [0,9459]

Chapter 3

Random Variable and Probability Distributions

3.1 Random Variable

A *probability distribution* or, briefly, *distribution*, shows the probabilities of events in an experiment. In probability theory and statistics, a *random variable* (or *stochastic variable*) is a variable whose value is subject to variations due to chance (i.e. randomness, in a mathematical sense). A random variable can take on a set of possible different values (similarly to other mathematical variables), each with an associated probability, in contrast to other mathematical variables.

Random variable values may represent the possible results of an experiment that has not yet been performed, or the possible results of a previous experiment whose existing value is uncertain (for example, due to incomplete information or inaccurate measurements). A mathematical function describing possible values of a random variable and their associated probabilities is called probability distribution.

In elementary probability theory a random variable we understand each mapping $X : \gamma \rightarrow \mathbb{R}$, where γ is the set of elementary events of a probability space $[\gamma, \tau, P]$. For each $E \in \gamma$ is $X(E)$ some real number, called *a value of a random variable X for the event E* .

Random variables will be denoted by capital letters X, Y, X_1, X_2, \dots and values of random variables will be denoted by small letters a, b, c, x, y, \dots . We suppose that for each $a \in \mathbb{R}$ we can determine the probabilities of types:

1. $P(X = a)$, i. e., the probability that the value of the random variable X is equal to the number a ;
2. $P(X \leq a)$, i. e., the probability that the value of the random variable X is not greater than the number a ;

3. $P(X \in I)$, i. e., the probability that the value of the random variable X take values from the interval I .

Random variables can be

1. *discrete*, that is, taking any of a specified finite or countable list of values,
2. *continuous*, taking any numerical value in an interval or collection of intervals.

Examples of a discrete random variable are the sum of the thrown values on two dice, the number of points obtained in the credit test, the number of correct test responses, the number of white balls withdrawn from the urn containing the black and white balls, etc.

Continuous random variables are, for example, the half-life of a radioactive substance, the height of an adult woman, the density of a chemical, the elasticity of a material, etc.

3.2 Cumulative Distribution Function

Definition 3.1. *Cumulative Distribution Function* (or briefly *Distribution Function*) of random variable X is the function defined for each $x \in \mathbb{R}$ as follows

$$F(x) = P(X \leq x). \tag{3.1}$$

Theorem 3.1 (Properties of the distribution function).

1. $0 \leq F(x) \leq 1$ for each $x \in \mathbb{R}$;
2. $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$;
3. F is non-decreasing function, i. e., if $a < b$ then $F(a) \leq F(b)$;
4. F is right continuous on the entire set of real numbers;
5. if $a < b$, then $P(X \in (a, b]) = P(a < X \leq b) = F(b) - F(a)$.

Discrete random variables and distributions

Definition 3.2. A random variable X has a distribution of discrete type if X takes on only finitely many or at most countably many values x_1, x_2, x_3, \dots called the *possible values* of X , with probabilities $p_i = P(X = x_i)$ whereas the probability $P(X \in I)$ is zero for any interval I containing no possible values.

Obviously, the discrete distribution is also determined by the *probability function* $f(x)$ of X , defined by

$$f(x) = \begin{cases} p_j & \text{if } x = x_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

The set $\mathcal{H}(X) = \{x_1, x_2, x_3, \dots, x_n\}$ is called a *range of values* of X .

We can describe a random variable X by the table of the form

$$\frac{x_i}{P(X = x_i) = p_i} \parallel \begin{array}{c|c|c|c} x_1 & x_2 & \cdots & x_n \\ \hline p_1 & p_2 & \cdots & p_n \end{array}, \quad (3.3)$$

which is called *probability table of random variable X* . It is easy to see, that for the probabilities so-called *normalization condition* holds:

$$\sum_{i=1}^n p_i = 1. \quad (3.4)$$

We get the values of distribution function $F(x)$ by taking sums,

$$F(x) = \sum_{x_j \leq x} f(x_j) = \sum_{x_j \leq x} p_j. \quad (3.5)$$

Continuous random variables and density function

Definition 3.3. A random variable X is called *continuous*, if there exists a non-negative and on the set \mathbb{R} integrable function f such that the following assertions hold:

$$F(x) = \int_{-\infty}^x f(t) dt, \quad x \in (-\infty, \infty), \quad (3.6)$$

where F is the distribution function of variable X .

Function f satisfying (3.6) is called a *probability density function* of random variable X .

Theorem 3.2 (Properties of probability density function). Let F be the distribution function and let f be the density function. Then the following assertions hold:

1. if $f(x)$ exists, then $f(x) \geq 0$;
2. if there exists $F'(x)$, then $F'(x) = f(x)$, $x \in \mathbb{R}$;
3. the normalization condition for the density function is

$$\int_{-\infty}^{\infty} f(x) dx = 1; \quad (3.7)$$

4. for $a < b$ we have

$$P(a \leq X \leq b) = \int_a^b f(x) dx = F(b) - F(a), \quad (3.8)$$

i. e., the probability that a random variable X takes values from $\langle a, b \rangle$ is equal to the content of the area, which is bounded by the graph of the density over this interval (see Figure 3.1).

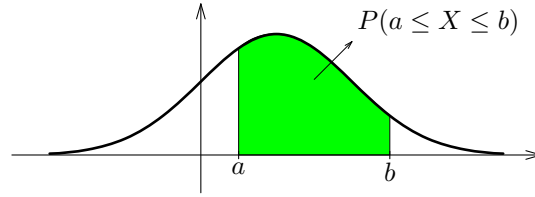


Figure 3.1: Example illustrating formula (3.8).

Remark 3.1. Additionally, for a continuous random variable is valid:

1. the density function of a random variable X is not determined uniquely;
2. the distribution function is continuous on \mathbb{R} ;
3. for each $a \in \mathbb{R}$ is $P(X = a) = 0$;
4. we can generalize the equality (3.8) as follows:

$$\begin{aligned}
 P(a < X < b) &= P(a \leq X < b) = P(a < X \leq b) = \\
 &= P(a \leq X \leq b) = \int_a^b f(x) dx.
 \end{aligned}
 \tag{3.9}$$

3.3 Numerical Parameters of Random Variable

For each random variable we assign so-called *numerical parameters*, which will give us some information on character of the studied random variable. In the following we will suppose that the probability distribution of random variable X is given by the density function f , if X is continuous, and by the probability Table (3.4), if X is discrete.

Position Parameters

Definition 3.4 (Expected value). Let the law of the probability distribution of random variable X be given. An *expected value of random variable X* is the number $E(X)$ defined as follows

$$E(X) = \begin{cases} \sum_{i=1}^{n(\infty)} x_i p_i & \text{for discrete random variable,} \\ \int_{-\infty}^{\infty} x f(x) dx & \text{for continuous random variable.} \end{cases}
 \tag{3.10}$$

Theorem 3.3 (Properties of the expected value). Let X and Y be random variables and let a and b be real constants. Then

1. the expected value of the constant random variable A taking value a is a , i. e., $E(A) = a$;

2. $E(a \cdot X + b \cdot Y) = a \cdot E(X) + b \cdot E(Y)$;
3. $E(X - E(X)) = 0$;
4. if the density graph of a continuous random variable X is symmetric with respect to the line $x = a$ then $E(X) = a$;
5. if $Z = g(X)$ then

$$E(Z) = E(g(X)) = \begin{cases} \sum_{i=1}^{n(\infty)} g(x_i) p_i & \text{for discrete random variable,} \\ \int_{-\infty}^{\infty} g(x) f(x) dx & \text{for continuous random variable.} \end{cases} \quad (3.11)$$

Definition 3.5. Other position parameters are:

- a) The *mode of a random variable* X (we denote $\mathcal{M}o(X)$), which is defined for a discrete random variable as the most probable value of X and for a continuous random variable as any point at which the density function shall enter into a local maximum.
- b) The *median of a random variable* X (we denote $\mathcal{M}e(X)$), which is defined as the number for which $P(X \leq \mathcal{M}e(X)) \geq 0,5$ while $P(X \geq \mathcal{M}e(X)) \geq 0,5$.

Parameters of Variance

Definition 3.6. Let the law of the probability distribution of random variable X with the expected value $E(X)$ be given. The *variance of random variable* X is the number $D(X)$ defined as follows

$$D(X) = \begin{cases} \sum_{i=1}^{n(\infty)} (x_i - E(X))^2 p_i & \text{for discrete random variable,} \\ \int_{-\infty}^{\infty} (x - E(X))^2 f(x) dx & \text{for continuous random variable.} \end{cases} \quad (3.12)$$

Remark 3.2. For $g(x) = (x - E(X))^2$ we get from (3.11) and (3.12)

$$D(X) = E[(X - E(X))^2]. \quad (3.13)$$

Theorem 3.4 (Properties of variance). Let X be a random variable and let a a b be any constants. Then

1. the variance of the constant random variable A is $D(A) = 0$;
2. $D(a \cdot X) = a^2 \cdot D(X)$;

- 3. $D(a \cdot X + b) = a^2 \cdot D(X)$;
- 4. $D(X)$ can be expressed in the form

$$D(X) = E(X^2) - [E(X)]^2. \tag{3.14}$$

Definition 3.7. Let there exists the variance $D(X)$ of random variable X . The *standard deviation of random variable X* is the number $\sigma(X)$ defined as follows

$$\sigma(X) = \sqrt{D(X)}. \tag{3.15}$$

Definition 3.8. Random variable X is *standard random variable* if

$$E(X) = 0 \quad \text{and} \quad D(X) = 1. \tag{3.16}$$

Theorem 3.5 (Standard random variable). Let X be a random variable with the expected value $E(X)$ and $D(X) \neq 0$. The random variable

$$Y = \frac{X - E(X)}{\sigma(X)} \tag{3.17}$$

is standard random variable, i. e., $E(Y) = 0$ and $D(Y) = 1$.

3.4 Probability Distributions of Discrete Random Variables

3.4.1 Binomial distribution

The *binomial distribution* occurs in games of chance (rolling a dice, tossing a coin), quality inspection (e.g. count of the number of defects), opinion polls (number of employees favouring certain schedule changes, etc.), medicine (number of patients covered by a new medication), and so on. The condition of its occurrence is as follows.

We are interested in the number of times an event A occurs in n independent trials. In each trial the event A has the same probability of occurrence $P(A) = p$. Then in a trial, A will not occur with probability $q = 1 - p$. In n trials, the random variable that interests us is

$$X = \text{Number of times } A \text{ occurs in } n \text{ trials.}$$

X can assume the values $0, 1, 2, \dots, n$, and we want to determine the corresponding probabilities. Now $X = x$ means that A occurs in x trials and in $n - x$ trials does not occur. Let $B = \bar{A}$ be the complement of A , meaning that A does not occur. The probability

that A occurs in first x trials and does not occur in the remaining $n - x$ trials is $p^x \cdot q^{n-x}$. Now, it is just one order of arranging x A 's and $n - x$ B 's. The number of orders of arranging x A 's and $n - x$ B 's is the number of combinations of x -th class of n elements, i. e., $\binom{n}{x}$. Then the probability that A occurs in x trials and does not occur in $n - x$ trials is $\binom{n}{x} \cdot p^x \cdot (1 - p)^{n-x}$.

Definition 3.9. A random variable X has a *binomial distribution with parameters n and p* if

1. its range of values is $\mathcal{H}(X) = \{0, 1, 2, \dots, n\}$;
2. $P(X = x) = \binom{n}{x} \cdot p^x \cdot (1 - p)^{n-x}$ for each $x \in \{0, 1, \dots, n\}$.

We shall use the notation $X \sim \text{bino}(n; p)$.

Theorem 3.6. If $X \sim \text{bino}(n; p)$, then

$$E(X) = n \cdot p, \quad D(X) = n \cdot p \cdot q \quad \text{and} \quad \sigma(X) = \sqrt{n \cdot p \cdot q}. \quad (3.18)$$

Moreover,

$$\mathcal{M}o(X) = k_0 \in \langle np - q, np + p \rangle. \quad (3.19)$$

Proof. We prove that $E(X) = n \cdot p$. We have

$$\begin{aligned} E(X) &= \sum_{i=0}^n x_i p_i = \sum_{x=0}^n x \cdot \binom{n}{x} \cdot p^x q^{n-x} = \\ &= \sum_{x=1}^n \frac{n \cdot (n-1) \cdot \dots \cdot (n-x+1)}{x \cdot (x-1) \cdot \dots \cdot 2 \cdot 1} \cdot p^x q^{n-x}. \end{aligned}$$

Using the substitution $t = x - 1$ and binomial theorem we get

$$E(X) = np \sum_{t=0}^{n-1} x \binom{n-1}{t} p^t q^{(n-1)-t} = np(p+q)^{n-1} = n \cdot p. \quad \square$$

3.4.2 Hypergeometric Distribution

This probability distribution can be characterized by the model in which the set of M objects is given, whereby K of them has a certain property and $M - K$ does not have this property. From this file are without returning randomly chosen N objects. We want to calculate the probability that between selected objects just x objects have the property under consideration.

Definition 3.10. A random variable X has *hypergeometric probability distribution* with parameters M , K and N if

1. range of values is $\mathcal{H}(X) = \{\max\{0, K - M + N\}, \dots, \min\{K, N\}\}$;
2. $P(X = x) = \frac{\binom{K}{x} \binom{M - K}{N - x}}{\binom{M}{N}}$ for each $x \in \mathcal{H}(X)$.

We shall use the notation $X \sim \text{hyge}(M, K, N)$.

Theorem 3.7. If $X \sim \text{hyge}(M, K, N)$, then

$$E(X) = N \cdot \frac{K}{M} \quad \text{and} \quad D(X) = \frac{(M - N) \cdot N \cdot K}{(M - 1) \cdot M} \left(1 - \frac{K}{M}\right). \quad (3.20)$$

3.4.3 Poisson distribution

The *Poisson distribution* is a discrete probability distribution of a random variable X that has these characteristics:

- The experiment consists of counting the number of times x , and event occurs in a given interval. The interval can be an interval of time, space, area, or volume.
- The probability of the event occurring is the same for each interval (of time, space, area, or volume).
- The number of occurrences of the event in one interval is independent of the number of occurrences in other intervals.
- The mean number of successes, denoted λ , is known over the interval. That is, λ is the expected value over the given interval.

Definition 3.11. A random variable X has the *Poisson distribution with parameter λ* if and only if

1. its range of values is $\mathcal{H}(X) = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$;
2. the probability function is

$$f(x) = P(X = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!} \quad \text{for each } x \in \mathcal{H}(X). \quad (3.21)$$

We denote $X \sim \text{poiss}(\lambda)$.

Poisson distribution is distribution with infinitely many possible values. We show that the normalization condition is satisfied.

We have

$$\sum p_i = \sum_{x=0}^{\infty} \frac{\lambda^x \cdot e^{-\lambda}}{x!} = e^{-\lambda} \cdot \left(\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \right) = e^{-\lambda} \cdot e^{\lambda} = 1,$$

since $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$ is the Taylor expansion of function e^{λ} .

Theorem 3.8. If $X \sim \text{poiss}(\lambda)$, then

$$E(X) = \lambda, \quad D(X) = \lambda \quad \text{and} \quad \sigma(X) = \sqrt{\lambda}. \quad (3.22)$$

Proof. We prove that $E(X) = \lambda$:

$$E(X) = \sum_{i=0}^{\infty} x_i p_i = \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x \cdot e^{-\lambda}}{x!} = e^{-\lambda} \cdot \sum_{x=1}^{\infty} \frac{x \cdot \lambda \cdot \lambda^{x-1}}{x \cdot (x-1) \cdot \dots \cdot 2 \cdot 1}.$$

We substitute $x = t + 1$ and get

$$E(X) = \lambda e^{-\lambda} \sum_{t=0}^{\infty} \frac{\lambda^t}{t!} = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda.$$

3.5 Probability Distributions of Continuous Random Variables

3.5.1 Continuous Uniform Probability Distribution

Definition 3.12. A random variable X has the *continuous uniform probability distribution on the interval $\langle a, b \rangle$* , if the density function f is determined by

$$f(x) = \begin{cases} h & \text{for } x \in \langle a, b \rangle, \\ 0 & \text{for } x \notin \langle a, b \rangle, \end{cases} \quad (3.23)$$

for some $h \in \mathbb{R}$. We denote this distribution function by $X \sim \text{unif}(a; b)$.

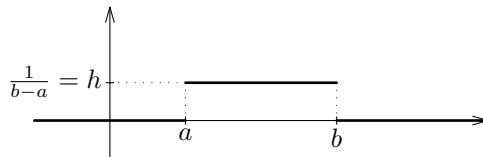


Figure 3.2 Density function of uniform distribution.

The graph of the density function (3.23) is shown on Figure 3.2. It is easy to see that $0 < h = \frac{1}{b-a}$. We find formula for the distribution function F : from (3.6) we have

$$F(x) = \int_{-\infty}^x 0 dt = 0 \quad \text{for } x \in (-\infty, a);$$

$$F(x) = \int_{-\infty}^a 0 dt + \int_a^x \frac{1}{b-a} dt = \frac{x-a}{b-a} \quad \text{for } x \in \langle a, b \rangle;$$

$$F(x) = \int_{-\infty}^a 0 dt + \int_a^b \frac{1}{b-a} dt + \int_b^x 0 dt = 1 \quad \text{for } x \in (b, \infty).$$

The graph of the distribution function is shown on Figure 3.3.

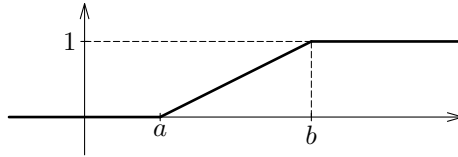


Figure 3.3 Distribution function of uniform distribution.

Continuous uniform probability distribution has a random variable that takes its values only at specific intervals of finite lengths, and the probability that its value will occur in any subintervals of that interval, is directly proportional to the length of the subinterval. This fact tends to formulate the following: all the values of a random variable from a given interval are equally likely.

Theorem 3.9. If $X \sim \text{unif}(a, b)$, then

$$E(X) = \frac{a+b}{2}, \quad D(X) = \frac{(b-a)^2}{12} \quad \text{and} \quad \sigma(X) = \frac{b-a}{2 \cdot \sqrt{3}} \approx 0,2887 \cdot (b-a). \quad (3.24)$$

3.5.2 Exponential Distribution

Definition 3.13. A random variable X has the *exponential distribution with parameter λ* if the density function f is

$$f(x) = \begin{cases} 1/\lambda e^{-x/\lambda} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases} \quad (3.25)$$

We denote the exponential distribution by $X \sim \text{exp}(\lambda)$.

The graph of (3.25) is shown on Figure 3.4. We prove the normalization condition (3.7):

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 0 dx + \int_0^{\infty} 1/\lambda \cdot e^{-x/\lambda} dx = 0 + \frac{1}{\lambda} \left[\frac{-e^{-x/\lambda}}{-1/\lambda} \right]_0^{\infty} = 1,$$

because $\lambda > 0$.

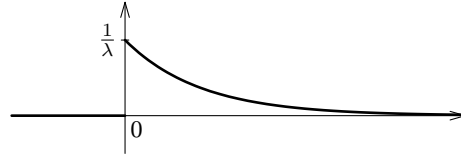


Figure 3.4 Density function of exponential distribution.

It is easy to see that $F(x) = 0$ for $x < 0$, for $x \geq 0$ we obtain

$$F(x) = \int_0^x 1/\lambda \cdot e^{-t/\lambda} dt = \left[-e^{-t/\lambda} \right]_0^x = 1 - e^{-x/\lambda}.$$

Hence the distribution function is

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 - e^{-x/\lambda} & \text{for } x \geq 0. \end{cases} \quad (3.26)$$

The graph is shown on Figure 3.5.

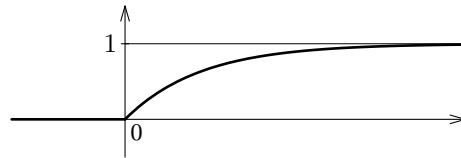


Figure 3.5 Distribution function of exponential distribution.

Theorem 3.10. If $X \sim \exp(\lambda)$, then

$$E(X) = \lambda, \quad D(X) = \lambda^2 \quad \text{and} \quad \sigma(X) = \lambda. \quad (3.27)$$

Remark 3.3.

1. With the exponential probability distribution is closely related to the durability of the issue, determining the warranty period for the product.
2. According to (3.27) is $E(X) = \lambda > 0$. Using (3.8) and (3.26) we get:
 $P(E(X) \leq X) = P(\lambda \leq X < \infty) = \lim_{x \rightarrow \infty} F(x) - F(\lambda) = 0 - (1 - e^{-1}) = e^{-1} \approx 0,3679$,
 i. e., the probability that random variable with exponential distribution will not be less than its mean value is always equal to the constant e^{-1} .

3.5.3 Normal Distribution

Definition 3.14. A random variable X has the *normal (the Gauss) distribution with parameters μ and $\sigma > 0$* if the density function f is determined by regulation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for each } x \in \mathbb{R}. \quad (3.28)$$

We denote $X \sim \text{norm}(\mu, \sigma)$ or $X \sim N(\mu, \sigma)$.

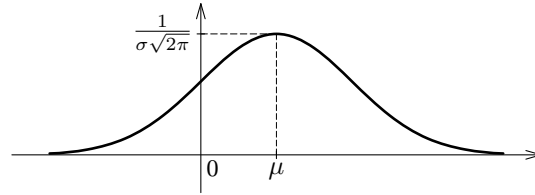


Figure 3.6 Density function of normal distribution.

The graph of the density function (3.28) is shown on Figure 3.6. It is easy to see that $f(x) \geq 0$, but the proof of the normalization condition (3.7)

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1 \quad \text{for each } \mu \in \mathbb{R} \text{ and } \sigma > 0 \quad (3.29)$$

is not trivial. The distribution function is

$$F(x) = P(X \leq x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \quad \text{for each } x \in \mathbb{R}. \quad (3.30)$$

The graph is shown on Figure 3.7.

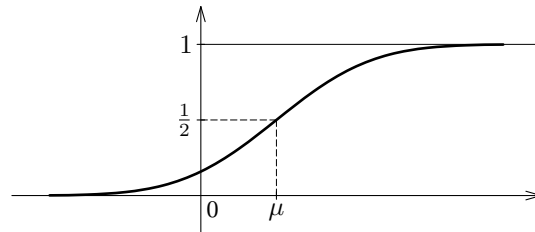


Figure 3.7 Distribution function of normal distribution.

Theorem 3.11. If $X \sim \text{norm}(\mu, \sigma)$, then

$$E(X) = \mu, \quad D(X) = \sigma^2 \quad \text{and} \quad \sigma(X) = \sigma. \quad (3.31)$$

By scaling of the random variable $X \sim \text{norm}(\mu, \sigma)$ we get the random variable

$$Y = \frac{X - \mu}{\sigma},$$

such that $E(Y) = 0$ and $D(Y) = 1$, so $Y \sim \text{norm}(0, 1)$. The density function φ of random variable Y is given by

$$\varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad \text{for each } y \in \mathbb{R} \quad (3.32)$$

and the distribution function Φ by

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{u^2}{2}} du \quad \text{for each } y \in \mathbb{R}. \quad (3.33)$$

The function φ is even. That means that

$$\Phi(-y) = 1 - \Phi(y) \quad \text{for each } y \in \mathbb{R}, \quad \left(\text{specially } \Phi(0) = \frac{1}{2} \right). \quad (3.34)$$

Using the substitution $u = t - \mu/\sigma$ we can (3.30) write in the form

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{u^2}{2}} du = \Phi\left(\frac{x-\mu}{\sigma}\right). \quad (3.35)$$

Theorem 3.12. If $X \sim \text{norm}(\mu, \sigma)$, then for each $a < b$ we have

$$\begin{aligned} P(a < X \leq b) &= P(a \leq X \leq b) = P(a \leq X < b) = \\ &= P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \end{aligned} \quad (3.36)$$

Theorem 3.13. If $X \sim \text{norm}(\mu, \sigma)$, then for each $\varepsilon > 0$ holds

$$P(|X - \mu| < \varepsilon) = P(\mu - \varepsilon < X < \mu + \varepsilon) = 2 \cdot \Phi\left(\frac{\varepsilon}{\sigma}\right) - 1. \quad (3.37)$$

Specially, for $\varepsilon = 3\sigma$ we get

$$P(|X - \mu| < 3\sigma) = P(\mu - 3\sigma < X < \mu + 3\sigma) = 2 \cdot \Phi(3) - 1 \approx 0,9973. \quad (3.38)$$

Proof. Using the previous theorem for $a = \mu - \varepsilon$ and $b = \mu + \varepsilon$ we have

$$\begin{aligned} P(\mu - \varepsilon < X < \mu + \varepsilon) &= \\ &= \Phi\left(\frac{\mu + \varepsilon - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - \varepsilon - \mu}{\sigma}\right) = \Phi\left(\frac{\varepsilon}{\sigma}\right) - \Phi\left(-\frac{\varepsilon}{\sigma}\right). \end{aligned}$$

If we use (3.34) for $y = \varepsilon/\sigma$, we get

$$P(\mu - \varepsilon < X < \mu + \varepsilon) = \Phi\left(\frac{\varepsilon}{\sigma}\right) - \left[1 - \Phi\left(\frac{\varepsilon}{\sigma}\right)\right] = 2 \cdot \Phi\left(\frac{\varepsilon}{\sigma}\right) - 1,$$

what is (3.37). Inequality (3.38) is trivial.

Remark 3.4. In practice, this property is usually formulated as so-called *three sigma rule*: Almost all values of random variable $X \sim \text{norm}(\mu, \sigma)$ (more precisely 99,73 %) are in the interval $\mu \pm 3\sigma$.

Suppose that $Y \sim \text{norm}(0, 1)$ with distribution function (3.33). Let $\alpha \in (0, 1)$ be a given real number. We are looking for $k_\alpha > 0$ such that

$$P(|Y| > k_\alpha) = \alpha. \tag{3.39}$$

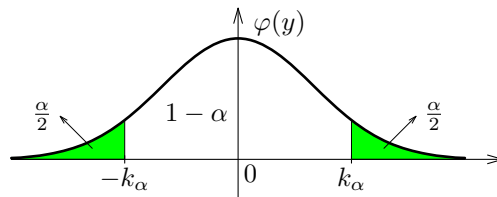


Figure 3.8 The graphic significance of equality (3.39).

Using the complementary event we can write (3.39) in the form

$$P(-k_\alpha \leq Y \leq k_\alpha) = 1 - \alpha. \tag{3.40}$$

Since Y is a continuous random variable, from (3.37) we get

$$P(-k_\alpha \leq Y \leq k_\alpha) = 2 \cdot \Phi(k_\alpha) - 1.$$

From the last two equalities we have $1 - \alpha = 2 \cdot \Phi(k_\alpha) - 1$ and consequently $\Phi(k_\alpha) = 1 - \frac{\alpha}{2}$, which means that (see Figure 3.8)

$$k_\alpha = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right), \tag{3.41}$$

where Φ^{-1} is inverse function of the distribution function (3.33). In mathematical statistics is k_α denoted as follows:

$$k_\alpha = y_{1-\frac{\alpha}{2}}. \tag{3.42}$$

3.6 Solved Examples

Example 3.1. The target consists of circle K and two rings K_1 and K_2 . Hit to circle K is obtained with a probability of 0,7 and is rewarded 15 points, for rings K_1, K_2 are probabilities $p_{K_1} = 0,2$ and $p_{K_2} = 0,1$, with reward 5 and -5 points, respectively. Let X be a random variable which takes the value of the sum of points achieved in two independent shots on target. We determine:

- a) the law of the probability distribution of a random variable X ;

- b) the expected value $E(X)$;
 c) the variance $D(X)$;
 d) $P(X \geq \frac{E(X)}{2})$.

Solution:

- a) It is easy to see that X is a discrete random variable with $\mathcal{H}(X) = \{-10, 0, 10, 20, 30\}$. We calculate the probabilities of individual values: e.g. value 0 is obtained when we achieve 5 points in the first shot and -5 points in the second shot, or in the reverse order:

$$P(X = 0) = 0,2 \cdot 0,1 + 0,1 \cdot 0,2 = 0,04;$$

Value 10 is obtained, when we achieve 5 points in each shot or 15 points in the first shot and -5 points in the second shot or -5 points in the first shot and 15 points in the second shot. We get

$$P(X = 10) = 0,7 \cdot 0,1 + 0,1 \cdot 0,7 + 0,2 \cdot 0,2 = 0,18.$$

In the same manner we obtain probabilities of other possible values. We obtain the following probability table of random variable X :

$$\begin{array}{c|c|c|c|c|c} x_i & -10 & 0 & 10 & 20 & 30 \\ \hline p_i = P(X = x_i) & 0,01 & 0,04 & 0,18 & 0,28 & 0,49 \end{array}. \quad (3.43)$$

- b) According to (3.10) we get

$$E(X) = (-10) \cdot 0,01 + 0 \cdot 0,04 + 10 \cdot 0,18 + 20 \cdot 0,28 + 30 \cdot 0,49 = 22.$$

- c) By (3.12) we obtain

$$\begin{aligned} D(X) &= (-10 - 22)^2 \cdot 0,01 + (0 - 22)^2 \cdot 0,04 + (10 - 22)^2 \cdot 0,18 + \\ &\quad + (20 - 22)^2 \cdot 0,28 + (30 - 22)^2 \cdot 0,49 = 88. \end{aligned}$$

- d) From $\mathcal{H}(X)$ and part b) we have

$$\begin{aligned} P\left(X \geq \frac{E(X)}{2}\right) &= P(X \geq 11) = P(X \in \{20, 30\}) = \\ &= P(X = 20) + P(X = 30) = 0,28 + 0,49 = 0,77. \end{aligned}$$

✓

Example 3.2. The student passes examinations at the ordinary term with probability 0,7 and event of failure, the probability of passing the test by him on the resit exam increases still by 0,1. There are two resists. We determine formula and draw a graph of the distribution function of a random variable that takes the value of completed terms on student test.

Solution:

The student pass exam to the ordinary term with probability 0,7 so $P(X = 1) = 0,7$. The student completes two terms, if student does not pass exam to the ordinary term, but he pass term in the first resist which implies that $P(X = 2) = 0,3 \cdot 0,8 = 0,24$. The student will participate in three tests in the event that on the first two terms he was unsuccessful, i. e., $P(X = 3) = 0,3 \cdot 0,2 = 0,06$. We obtain the probability table of random variable X :

$$\begin{array}{c|c|c|c} x_i & 1 & 2 & 3 \\ \hline p_i & 0,7 & 0,24 & 0,06 \end{array} .$$

By (3.1), the distribution function of random variable X has formula:

$$F(x) = \begin{cases} 0 & \text{for } x < 1, \\ 0,7 & \text{for } 1 \leq x < 2, \\ 0,94 & \text{for } 2 \leq x < 3, \\ 1 & \text{for } 3 \leq x. \end{cases}$$

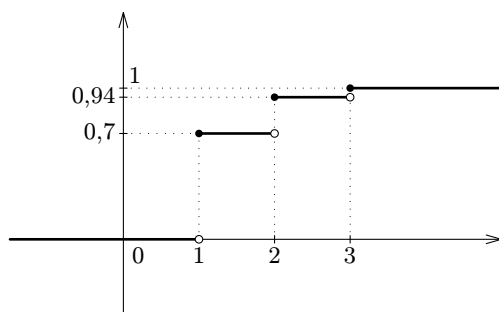


Figure 3.9 Distribution function.

✓

Example 3.3. Let the function $f(x) = \begin{cases} ax^{-3} & \text{for } x > 2, \\ 0 & \text{for } x \leq 2 \end{cases}$ be given. We determine:

- the number $a \in \mathbb{R}$ such that $f(x)$ is the density function of some random variable X ;
- the expected value $E(X)$;
- $P(1 \leq X < E(X))$.

Solution:

a) We shall use (3.7):

$$1 = \int_{-\infty}^{\infty} f(x) dx = \int_2^{\infty} \frac{a}{x^3} dx = \left[\frac{-a}{2x^2} \right]_2^{\infty} = \frac{a}{8},$$

which implies $a = 8$.

b) By (3.10) and (3.12) we obtain:

$$E(X) = \int_{-\infty}^{\infty} xf(x) dx = \int_2^{\infty} x \cdot \frac{8}{x^3} dx = \left[\frac{-8}{x} \right]_2^{\infty} = 4.$$

c)

$$P(1 \leq X < E(X)) = \int_1^4 f(x) dx = \int_2^4 \frac{8}{x^3} dx = \left[\frac{-4}{x^2} \right]_2^4 = \frac{3}{4}.$$

✓

Example 3.4. Let us have the function $F(x) = \begin{cases} a & \text{for } x < 1, \\ bx + cx^2 & \text{for } 1 \leq x < 3, \\ d & \text{for } 3 \leq x, \end{cases}$

where a, b, c, d are real constants. We determine:

- the values of a, b, c, d , such that F can be a distribution function of some random variable X ;
- the density function f of those random variable;
- $P(0 < X \leq 2)$.

Solution:

a) For the distribution function holds

$$\lim_{x \rightarrow -\infty} F(x) = a \quad \text{and} \quad \lim_{x \rightarrow \infty} F(x) = d$$

and by Theorem 3.1 we get $a = 0$ and $d = 1$. Since F is not constant on interval $\langle 1, 3 \rangle$, X is a continuous random variable. This implies that F is a continuous function on \mathbb{R} , so F has to be continuous at points 1 and 3. Since

$$\lim_{x \rightarrow 1^-} F(x) = 0, \quad \lim_{x \rightarrow 1^+} F(x) = b + c,$$

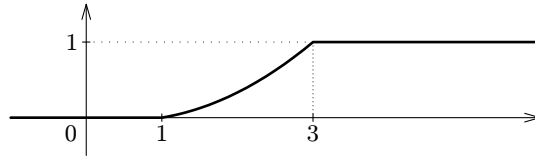


Figure 3.10: The distribution function of X .

$$\lim_{x \rightarrow 3^-} F(x) = 3b + 9c, \quad \lim_{x \rightarrow 3^+} F(x) = 1,$$

we obtain the system of linear equations $b + c = 0$, $3b + 9c = 1$. The solution is $b = -1/6$, $c = 1/6$. We obtain

$$F(x) = \begin{cases} 0 & \text{for } x < 1, \\ (-x + x^2)/6 & \text{for } 1 \leq x < 3, \\ 1 & \text{for } 3 \leq x, \end{cases}$$

which is the distribution function of some random variable X . The graph is shown on Figure 3.10.

b) From Theorem 3.2 we obtain the density function f :

$$f(x) = \begin{cases} 0 & \text{for } x < 1, \\ (2x - 1)/6 & \text{for } 1 \leq x < 3, \\ 0 & \text{for } 3 \leq x. \end{cases}$$

c) According to Remark 3.1 we have

$$P(0 < X \leq 2) = \int_0^2 f(x) dx = \int_0^1 0 dx + \int_1^2 \frac{2x - 1}{6} dx = \frac{1}{6} [x^2 - x]_1^2 = \frac{1}{3}.$$

We can obtain this result by (3.8), too:

$$P(0 < X \leq 2) = F(2) - F(0) = \frac{-2 + 2^2}{6} - 0 = \frac{1}{3}.$$

✓

Example 3.5. Let

$$f(x) = \begin{cases} k \cdot (x + 1) & \text{for } -1 < x < 0, \\ k & \text{for } 0 \leq x < 2, \\ 0 & \text{for } x \notin (-1, 2). \end{cases}$$

We determine:

- a) constant k , such that f can be a density function of some random variable X ;
- b) the distribution function of random variable X ;
- c) $P(X < 0)$.

Solution:

a) Using (3.7) we get

$$\int_{-\infty}^{-1} 0 \, dx + \int_{-1}^0 k(x+1) \, dx + \int_0^2 k \, dx + \int_2^{\infty} 0 \, dx = 1$$

$$\frac{k}{2} + 2k = 1 \Rightarrow k = 0,4.$$

We obtained

$$f(x) = \begin{cases} 0,4 \cdot (x+1) & \text{for } -1 < x < 0, \\ 0,4 & \text{for } 0 < x < 2, \\ 0 & \text{for } x \notin (-1, 2). \end{cases}$$

b) From (3.6) we obtain:

- for $x \leq -1$: $F(x) = \int_{-\infty}^x 0 \, dt = 0$;
- for $-1 < x \leq 0$: $F(x) = \int_{-\infty}^{-1} 0 \, dt + \int_{-1}^x 0,4 \cdot (t+1) \, dt = 0,2 \cdot (x+1)^2$;
- for $0 < x \leq 2$: $F(x) = \int_{-\infty}^{-1} 0 \, dt + \int_{-1}^0 0,4 \cdot (t+1) \, dt + \int_0^x 0,4 \, dt = 0,4x + 0,2$;
- for $2 \leq x$: $F(x) = \int_{-\infty}^{-1} 0 \, dt + \int_{-1}^0 0,4 \cdot (t+1) \, dt + \int_0^2 0,4 \, dt + \int_2^x 0 \, dt = 1$.

Hence

$$F(x) = \begin{cases} 0 & \text{for } x \leq -1; \\ 0,2 \cdot (x+1)^2 & \text{for } -1 < x \leq 0, \\ 0,2 \cdot (2x+1) & \text{for } 0 < x \leq 2, \\ 1 & \text{for } x > 2. \end{cases}$$

$$\text{c) } P(X < 0) = \int_{-\infty}^0 f(x) \cdot dx = \int_{-1}^0 0,4 \cdot (x+1) \cdot dx = 0,4 \cdot \left[\frac{x^2}{2} + x \right]_{-1}^0 = 0,2. \quad \checkmark$$

Example 3.6. In the world series, there are two baseball teams - American League team and National League team. The series ends when the winning team wins 4 games. We assume that the teams are evenly matched. Determine the probability that the world series will last

- a) 4 games,
- b) 5 or 6 games.

Solution:

This is a very tricky application of the binomial distribution. For the purpose of this analysis, we assume that the teams are evenly matched. Therefore, the probability that a particular team wins a particular game is 0,5. We define a success as a win by the team that ultimately becomes the world series champion.

a) This can occur if one team wins the first 4 games. The probability of the National League team winning 4 games in a row is:

$$P(X = 4) = \binom{4}{4} 0,5^4 \cdot 0,5^0 = 0,0625.$$

Similarly, when we compute the probability of the American League team winning 4 games in a row, we find that it is also 0,0625. Therefore, probability that the series ends in four games would be $0,0625 + 0,0625 = 0,125$; since the series would end if either the American or National League team won 4 games in a row.

b) Now let's tackle the question of finding probability that the world series ends in 5 games. The trick in finding this solution is to recognize that the series can only end in 5 games, if one team has won. So let's first find the probability that the American League team wins exactly 3 of the first 4 games.

$$P(X = 3) = \binom{4}{3} \cdot 0,5^3 \cdot 0,5^1 = 0,25.$$

Given that the American League team has won 3 of the first 4 games, the American League team has a 50% chance of winning the fifth game to end the series. Therefore, the probability of the American League team winning the series in 5 games is $0,25 \cdot 0,50 = 0,125$. Since the National League team could also win the series in 5 games, the probability that the series ends in 5 games would be

$$P(A) = 0,125 + 0,125 = 0,25.$$

The probability that the world series ends in 6 series would be solved in the same way. The probability that American League team win in 6 series is

$$0,5 \cdot P_{5,0,5}(3) = 0,5 \cdot \binom{5}{3} \cdot 0,5^3 \cdot 0,5^2 = 0,15625.$$

Since the National League team could also win the series in 6 games, the probability that the series ends in 5 games would be

$$P(B) = 0,15625 + 0,15625 = 0,3125.$$

Since the events A and B are disjoint the probability that the world series will last 5 or 6 games is

$$P(A \cup B) = 0,25 + 0,3125 = 0,5625.$$

✓

Example 3.7. Suppose we select 5 cards from an ordinary deck of playing cards. What is the probability of obtaining 2 or fewer hearts?

Solution:

This is a hypergeometric experiment in which we know the following:

$M = 52$, since there are 52 cards in a deck,

$k = 13$, since there are 13 hearts in a deck,

$N = 5$, since we randomly select 5 cards from the deck,

$x = 0$ to 2 , since our selection includes 0, 1, or 2 hearts.

We plug these values into the hypergeometric formula as follows:

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{\binom{13}{0} \binom{39}{5}}{\binom{52}{5}} + \frac{\binom{13}{1} \binom{39}{4}}{\binom{52}{5}} + \frac{\binom{13}{2} \binom{39}{3}}{\binom{52}{5}} = 0,9072.$$

✓

Example 3.8. Certain website is visited for the period for one hour in average by 30 guests. We determine:

- the probability that during four minutes visit this page one guest;
- the probability that during four minutes visit this page at least one guest;
- the probability that during four minutes visit this page at least three, but less than eleven guests;
- the probability that during ten minutes visit this page at most five guests;
- the probability that during ten minutes visit this page more than three guests.

Solution:

For tasks a) – c) we denote by X the random variable, which takes the value of the number of visitors of website during four minutes. The expected value of X is $E(X) = \lambda_1 = 4 \cdot 30/60 = 2$.

a) According to (3.21) we have

$$P(X = 1) = \frac{2^1 \cdot e^{-2}}{1!} \approx 0,2707.$$

b) We use the opposite event:

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{2^0 \cdot e^{-2}}{0!} \approx 0,8647.$$

c) We have to compute $P(3 \leq X < 11)$:

$$P(3 \leq X < 11) = \sum_{x=3}^{10} P(X = x) = \sum_{x=3}^{10} \frac{2^x \cdot e^{-2}}{x!} \approx 0,3233.$$

For tasks d) – e) we denote by Y the random variable, which takes the value of the number of visits of website within ten minutes. The expected value of Y is $E(Y) = \lambda_2 = \frac{30}{60} \cdot 10 = 5$.

d) We compute

$$P(Y \leq 5) = \sum_{y=0}^5 P(Y = y) = \sum_{y=0}^5 \frac{5^y \cdot e^{-5}}{y!} \approx 0,616.$$

e) We use the opposite event:

$$P(Y > 3) = 1 - P(Y \leq 3) = 1 - \sum_{y=0}^3 P(Y = y) = 1 - \sum_{y=0}^3 \frac{5^y \cdot e^{-5}}{y!} \approx 0,735.$$

✓

Example 3.9. The lifetime of the product has an exponential distribution with an expected value 200 hours. We determine:

- a) the probability that the product is functional at least 300 hours;
- b) the probability that the product will not be functional in excess of its average lifetime;
- c) the maximum warranty period to be guaranteed if the manufacturer allows a maximum of 5% of complaints about the product.

Solution:

Let T be the random variable, which takes the value of the product lifetime. T has an exponential distribution, so $E(T) = 200 = \lambda$ and $X \sim \exp(200)$.

a) We are required to compute $P(T \geq 300)$. We obtain

$$P(T \geq 300) = 1 - P(T < 300) = 1 - P(T \leq 300) = 1 - F(300) = 1 - (1 - e^{-1,5}) \approx 0,2231.$$

b) The average lifetime of the product is actually the expected value of the random variable T . We have

$$P(T \leq E(T)) = P(T \leq 200) = F(200) = 1 - e^{-1} \approx 0,6321.$$

c) It is necessary to determine such a maximum value z , that the inequality $P(T \leq z) \leq 0,05$ is satisfied, so $F(z) \leq 0,05$. We get

$$1 - e^{-\frac{z}{200}} \leq 0,05 \Rightarrow e^{-\frac{z}{200}} \geq 0,95 \Rightarrow -\frac{z}{200} \geq \ln 0,95 \Rightarrow z \leq -200 \cdot \ln 0,95 \approx 10,2587.$$

The manufacturer would probably give warranty for 10 hours.

✓

Example 3.10. Suppose that the amount of time one spends in a bank is exponentially distributed with mean 10 minutes.

- What is the probability that a customer will spend more than 15 minutes in the bank?
- What is the probability that a customer will spend more than 15 minutes in the bank given that he is still in the bank after 10 minutes?

Solution:

a) $P(X > 15) = 1 - P(X \leq 15) = 1 - F(15) = 1 - (1 - e^{-15/10}) = e^{-1,5} = 0,22.$

b) We have to compute the conditional probability $P(X > 15|X > 10)$. Since

$$P(X > 10) = 1 - F(10) = 1 - (1 - e^{-10/10}) = e^{-1},$$

we obtain

$$P(X > 15|X > 10) = \frac{P(X > 15 \wedge X > 10)}{P(X > 10)} = \frac{P(X > 15)}{P(X > 10)} = \frac{e^{-1,5}}{e^{-1}} = e^{-0,5} = 0,606.$$

✓

Example 3.11. 33 bulbs are connected in parallel into a circuit, and it is known that each of them is defective with probability 0,1.

- What is the probability that of these bulbs are more bulbs defective, than we could on average expect?
- What is the probability that of these bulbs are less bulbs defective than we can most likely expect?

Solution:

a) Let X be a random variable, which takes the value of the number of defective bulbs. Obviously $X \sim \text{bino}(33; 0,1)$. On average, we can expect $E(X) = n \cdot p = 33 \cdot 0,1 = 3,3$ defective bulbs. We obtain

$$\begin{aligned} P(X > E(X)) &= P(X > 3,3) = P(X \geq 4) = \\ &= \sum_{x=4}^{33} \binom{33}{x} 0,1^x \cdot 0,9^{33-x} \approx 0,4231. \end{aligned}$$

b) The number defective lamps that we can most likely expect is the modus of random variable X . According to (3.19) we have

$$\mathcal{M}o(X) \in \langle 33 \cdot 0,1 - 0,9; 33 \cdot 0,1 + 0,1 \rangle = \langle 2,4; 3,4 \rangle,$$

and therefore $\mathcal{M}o(X) = 3$. We have

$$P(X < \mathcal{M}o(X)) = P(X < 3) = P(X \leq 2) = \sum_{x=0}^2 \binom{33}{x} 0,1^x 0,9^{33-x} \approx 0,3457.$$

✓

Example 3.12. The mass of produced weights has a normal probability distribution with mean values of 10 grams. The manufacturer provides a standard deviation of 0,02 g. We determine the probability that randomly bought weights will have a real mass

- a) greater than 10,03 g;
- b) less than 9,99 g;
- c) at least 10 g but not more than 10,05 g.

Solution:

Let X be a random variable which takes the value of real mass of bought weights. Obviously $X \sim \text{norm}(10, 0, 02)$.

a) We want to determine $P(X > 10,03)$. We get

$$\begin{aligned} P(X > 10,03) &= 1 - P(X \leq 10,03) = 1 - F(10,03) = \\ &= 1 - \Phi\left(\frac{10,03 - 10}{0,02}\right) = 1 - \Phi(1,5) \approx 1 - 0,9332 = 0,0668. \end{aligned}$$

b) We are required to compute $P(X < 9,99)$. We have

$$P(X < 9,99) = P(X \leq 9,99) = F(9,99) = \Phi\left(\frac{9,99 - 10}{0,02}\right) = \Phi(-0,5) = 1 - \Phi(0,5) \approx 0,3085.$$

c) According to (3.36) we have

$$\begin{aligned} P(10 \leq X \leq 10,05) &= \Phi\left(\frac{10,05 - 10}{0,02}\right) - \Phi\left(\frac{10 - 10}{0,02}\right) = \\ &= \Phi(2,5) - \Phi(0) \approx 0,4938. \end{aligned}$$

✓

Example 3.13. Measuring with a voltmeter is loaded with 5 Volts systematic error and random errors have a normal probability distribution with a standard deviation of 2 Volts. We perform one measurement. What is the probability that the error of the measured values will differ by 1 volt from

- a) mean value of the expected error,
- b) actual measured values?
- c) What may be with probability 0,99 the maximum deviation of the measurement error from its mean value?

Solution:

Let X be the random variable, which takes the value of the error in one measurement by the voltmeter. The systematic error is in fact the average error, i. e., $\mu = 5$. Since $\sigma = 2$ then $X \sim \text{norm}(5, 2)$.

a) We compute $P(|X - 5| < 1)$. From (3.37) we get for $\varepsilon = 1$:

$$P(|X - 5| < 1) = 2 \cdot \Phi\left(\frac{1}{2}\right) - 1 \approx 0,3829.$$

b) We want to determine $P(|X| < 1)$. We have

$$P(|X| < 1) = P(-1 < X < 1) = F(1) - F(-1) \approx 0,0215.$$

where $F(1) = \text{normcdf}(1, 5, 2) \approx 0,0228$ and $F(-1) = \text{normcdf}(-1, 5, 2) \approx 0,0013$.

c) We want to determine the value ε , such that $P(|X - \mu| < \varepsilon) = 0,99$. From (3.37) we obtain

$$P(|X - \mu| < \varepsilon) = P(|X - 5| < \varepsilon) = 2 \cdot \Phi\left(\frac{\varepsilon}{2}\right) - 1 = 0,99.$$

Hence

$$\Phi\left(\frac{\varepsilon}{2}\right) = 0,995 \quad \text{which means that} \quad \varepsilon = 2 \cdot \Phi^{-1}(0,995) \approx 5,1517.$$

✓

3.7 Unsolved Tasks

3.1. Let

$$f(x) = \begin{cases} k \cdot (3x - x^2) & \text{for } 0 < x < 3, \\ 0 & \text{for } x \notin (0, 3). \end{cases}$$

Determine:

- constant k , such that f can be a density function of some random variable X ;
- the distribution function of random variable X ;
- $P(X < 0)$;
- $E(X)$ and $D(X)$.

3.2. Let

$$f(x) = \begin{cases} k/x^4 & \text{for } x \geq 1, \\ 0 & \text{for } x < 1. \end{cases}$$

Determine:

- constant k such that f can be a density function of some random variable X ;

- b) the distribution function of random variable X ;
- c) $E(X)$ and $D(X)$;
- d) $P(\sqrt[3]{2} < X < D(X))$.

3.3. Let

$$f(x) = \begin{cases} k \cdot \cos 2x & \text{for } x \in (0, \frac{\pi}{4}), \\ 0 & \text{for } x \notin (0, \frac{\pi}{4}). \end{cases}$$

Determine:

- a) constant k such that f can be a density function of some random variable X ;
- b) the distribution function of random variable X ;
- c) $E(X)$ and $D(X)$;
- d) $P(-3 < X < \frac{\pi}{12})$.

3.4. Let us have the function $F(x) = \begin{cases} 0 & \text{for } x \leq 3, \\ ax + b & \text{for } 3 < x \leq 6, \\ c & \text{for } x > 6. \end{cases}$, where a, b, c are real constants. Determine:

- a) the values of a, b, c such that F can be a distribution function of some random variable X ;
- b) the density function f of those random variable;
- c) $E(X)$ and $D(X)$;
- d) $P(4 < X \leq 5)$ and $P(-0,5 < X \leq 5)$.

3.5. Let us have the function $F(x) = \begin{cases} 0 & \text{for } x \leq -5, \\ a \cdot (x + b) & \text{for } -5 < x \leq 2, \\ 1 & \text{for } x > 2, \end{cases}$ where a, b are real constants. Determine:

- a) the values of a, b such that F can be a distribution function of some random variable X ;
- b) the density function f of those random variable;
- c) $P(-2 < X \leq 2)$ and $P(-6 < X \leq 1)$.

3.6. Let us have the function $F(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ a \cdot (x^2 - x^4/4) & \text{for } 0 < x \leq \sqrt{2}, \\ b & \text{for } x > \sqrt{2}, \end{cases}$

where a, b are real constants. Determine:

- the values of a, b such that F can be a distribution function of some random variable X ;
- the density function f of those random variable;
- $E(X)$ and $D(X)$.

3.7. Let us have the function $F(x) = \begin{cases} c & \text{for } x \leq 0, \\ a + b \cdot \sin x & \text{for } 0 < x \leq \frac{\pi}{2}, \\ d & \text{for } x > \frac{\pi}{2}. \end{cases}$,

where a, b, c, d are real constants. Determine:

- the values of a, b, c, d such that F can be a distribution function of some random variable X ;
- the density function f of those random variable;
- $E(X)$ and $D(X)$;
- $P(0 < X \leq \frac{\pi}{4})$ and $P(0 < X \leq \frac{\pi}{2})$.

3.8. Let us have the function $F(x) = \begin{cases} d & \text{for } x \leq 0, \\ a + b \cdot e^{-x} & \text{for } x > 0, \end{cases}$

where a, b are real constants. Determine:

- the values of a, b such that F can be a distribution function of some random variable X ;
- the density function f of those random variable;
- $P(1 < X \leq 4)$, $P(X \geq 2)$ and $P(0 < X \leq 3)$;
- $E(X)$ and $D(X)$.

3.9. Let us have the function

$$F(x) = a + b \cdot \operatorname{arctg} x$$

for $x \in (-\infty, \infty)$, where a, b are real constants. Determine:

- the values of a, b such that F can be a distribution function of some random variable X ;
- the density function f of those random variable;

- c) $P(\frac{1}{\sqrt{3}} < X \leq 1)$ and $P(-1 < X \leq \sqrt{3})$;
- d) $E(X)$ and $D(X)$.

3.10. Which of the following random variables are discrete and which are continuous?

- a) The number of students in a section of a statistics course.
- b) The air pressure in an car tire.
- c) The number of osprey chicks living in a nest.
- d) The height of students at TUKE.
- e) The speed of randomly selected vehicles on a highway.
- f) The time it takes a student to register for spring semester.

3.11. The number of industrial accidents at a particular plant is found to average 3 per month. Find the probability that

- a) exactly 6 accidents will occur at any given month,
- b) at least 2 accidents will occur at any given month,
- c) at most 7 accidents will occur at any given month.

3.12. One survey showed that 59% of Internet users are somewhat concerned about the privacy of their e-mail. Based on this information, what is the probability that for a random sample of 10 Internet users, 6 are concerned about their e-mail privacy?

3.13. For a certain section of a pine forest, the number of diseased trees per acre X follows the Poisson distribution with $\lambda = 10$. Find the probability that a randomly selected acre from this forest will contain

- a) at least 12 diseased trees,
- b) at most 8 diseased trees,
- c) more than 7 but less than 12 diseased trees.

3.14. Suppose that 65% of survey questionnaires sent to all faculty are completed and returned. If ten faculty members are chosen at random, compute the probability that

- a) exactly 6 will be completed and returned,
- b) less than 3 will be completed and returned,

c) more than 5 but less than 9 will be completed and returned.

3.15. A biologist studying a hybrid tomato found that there is a probability of 0,70 that the seeds will germinate. If the biologist plants 10 seeds, compute the probability that:

- a) exactly 8 seeds will germinate,
- b) at most 7 seeds will germinate,
- c) at least 8 seeds will germinate,
- d) between 3 and 7 seeds (inclusive, including 3 and 7) will germinate,
- e) between 4 and 9 seeds (exclusive) will germinate,
- f) less than 5 seeds will germinate,
- g) more than 3 seeds will germinate?

3.16. The playing life of a Sunshine radio is exponentially distributed with a mean of 600 hours. Find the probability that a randomly selected radio will last

- a) less than 800 hours,
- b) greater than 500 hours,
- c) between 600 and 700 hours.

3.17. A factory has a machine that puts corn flakes in boxes that are advertised as 200 grams each. If the distribution of weights is normal with $\mu = 200$ and $\sigma = 15$, what is the probability that the weight of a randomly selected box of corn flakes will be

- a) less than 195 grams,
- b) greater than 190 grams,
- c) between 180 and 210 grams.

3.18. Suppose you eat lunch at a restaurant that does not take reservations. Let X , representing the mean time waiting to be seated, has a normal distribution. It is known that the mean waiting time is 18 minutes with a standard deviation of 4 minutes. What is the probability that the waiting time will exceed 20 minutes given that it has exceeded 15 minutes?

3.19. Magic Video Games Inc. sells expensive computer games and wants to advertise an impressive, full-refund warranty period. It has found that the mean life for its computer games is 30 months with a standard deviation of 4 months. If the life spans of the computer games are normally distributed, how long of a warranty period (to the nearest month) can be offered so that the company will not have to refund the price of more than 7% of the computer games?

3.20. On average boat fishermen on Pyramid Lake catch 2 fish per 3 hour. Suppose you decide to fish the lake on a boat for 7 hours. In the 7-hour period of time what is the probability that you will catch

- a) 4 fishes,
- b) at least 3 fishes,
- c) between 3 and 6 fishes (inclusive)?

3.21. A loom which produces plaid wool fabric is known to produce, on the average, one noticeable flaw per 20 meters of fabric.

- a) What is the probability that there will be no flaws in a fifty-meters piece of the wool fabric?
- b) What is the probability that there will be more than 3 flaws in a 30-meters piece of fabric?
- c) What is the probability that there will be less than 5 flaws in a 100-meters piece of fabric?

3.22. The length of time a randomly chosen 9-year old child spends playing video games per day is approximately exponentially distributed with a mean equal to 2 hours.

- a) What is the probability that a randomly chosen 9-year old child will play video games at most 3 hours.
- b) What is the probability that a randomly chosen 9-year old child will play video games at least 4 hours.
- c) What is the probability that a randomly chosen 9-year old child will play video games between 1,5 hours and 3 hours.
- d) 70% of 9-year old children will play video games per day for at most how long?
- d) 60% of 9-year old children will play video games per day for at least how long?

3.23. Research has shown that studying improves a student's chances to 80% of selecting the correct answer to a multiple choice question. A multiple choice test has 15 questions. Each question has 4 choices.

- a) What is the probability that a student who studied answers at least 10 questions correctly?
- b) Suppose that a student does not study for the test but randomly guesses the answers. What is the probability that the student will answer 7 or 8 questions correctly?

3.24. When a customer calls the "Help Line" at ABC Computer Software Co., the amount of time that a customer must wait "on hold" until somebody answers the line and helps the customer follows an exponential distribution with mean of 7,5 minutes.

- a) What is the probability that a customer waits more than 10 minutes to receive help?
- b) What is the probability that a customer waits less than 8 minutes to receive help?
- c) 60% of customers waits for at least how long to receive help?
- d) 90% of customers waits for at most how long to receive help?

3.25. ABC Delivery Service offers next day delivery of packages weighing between 2 and 20 pounds in a certain city. They have found that the weights of the packages they deliver are uniformly distributed between 2 and 20 pounds.

- a) What is the probability that a package weighs between 10 and 15 pounds?
- b) Given that a package weighs less than 10 pounds, what is the probability that it weighs less than 5 pounds?
- c) 35% of packages weigh less than how many pounds?
- d) 50% of packages weigh more than how many pounds?

3.26. A wholesale warehouse supplies 25 shops. From each of them can independently from other shops to come during the day the order with probability 0,45. Determine:

- a) the expected value of the number of orders per day;
- b) the most probable number of orders per day, and the probability that this number of orders will come;
- c) the probability that during the day, at least 10 orders will come;
- d) the probability that during the day, at least five, but at most 15 orders will come.

3.27. Chance of hitting the atomic nucleus in accelerator in one experiment is 0,001.

- a) What is the probability that in the 5000 trials will hit the atomic nucleus of more than five times and less than ten times?
- b) Calculate the expected value, standard deviation and dispersion of nucleus hits.

3.28. Within one hour, on average 90 customers arrive at the fuel pump. Determine the probability that

- a) within 4 minutes will come exactly two customers;
- b) within 4 minutes will come at most two customers;
- c) within 4 minutes will come at least one customer.
- d) What is the smallest number of customers will be within 4 minutes not exceeded with a probability of at least 0,99?

3.29. Of the set of 80 products, between which there are 12 defective, a quality control randomly selects eight products. Determine the probability that between between the controlled products are

- a) five defective products;
- b) at least one defective product;
- c) less than five defective products.
- d) What number of defective products is the most probable?
- e) What is the average number of selected defective products?

3.30. Passengers can come to the tram stop at any moment. Determine:

- a) length of the interval between successive connections, if the probability that passengers will have to wait at least 4 minutes is 0,6;
- b) determine the expected value and standard deviation of the waiting time for a connection;
- c) the regulation of the distribution function of the random variable that takes the value of time waiting for connection.

3.31. Employees are coming to the terminal independently of one another by an average three per hour. Calculate the probability that

- a) during a three-hour interval come to the terminal at least 4 workers;
- b) during the two-hour period comes to a terminal more than five but less than eight workers.

[0.97877; 0.2983]

3.32. During the one hour comes to the server in average of 120 mails. Calculate the probability that

- a) in the nearest minute will come at least three mails,
- b) in the nearest 2 minutes will come at most three mails.

[0.3233; 0.4335]

3.33. In the time between 11 am and 13 pm arrives for lunch in the dining room on average 200 diners. We expect the number of diners is governed by the Poisson distribution. Calculate the probability that

- a) in the time between 11³⁰ and 11³² does not come any diner,
- b) within one minute come up to three diners,
- c) In the time between 11²⁰ and 11²³ come at least four diners.

[0.0357; 0.9117; 0.735]

3.34. The time at which a customer is served upon arrival at the store has an exponential distribution with a mean value of 4 minutes. What is the probability that

- a) the customer is served within 5 minutes after arriving at the store,
- b) the customer will not be served within 7 minutes after arriving at the store,
- c) the customer waits for service for more than 3 minutes but not more than 6 minutes.

[0.71350; 0.17377; 0.24924]

3.35. The lifetime of the electronic component has an exponential distribution with a mean value of 800 hours. Calculate the probability of that

1. the component will not crash within 1000 hours,
2. the component goes wrong in less than 600 hours,
3. the component goes wrong later than 500 hours, but less than 800 hours.

[0.2865; 0.5263; 0.16746]

3.36. The average time error-free operation of the product is 10,000 hours, and we estimate that this period is governed by the exponential distribution. Calculate the probability that

- a) product breaks down in less than 2,000 hours,
- b) the product will function correctly for more than 12,000 hours,

- c) the product will function correctly for more than 8,000 but less than 13,000 hours.
 [0.1813; 0.3012; 0.1768]

3.37. They come to the store for an hour on average 90 customers. The number of customers for a period of time is governed by the Poisson distribution. Calculate the probability that

- a) they will come any customers during 5 minutes,
 b) they will come at least three customers during 3 minutes.

[$5.53 \cdot 10^{-4}$; 0.82641]

3.38. Suppose we throw two dice at the same time. Let's make 3 throws. Let X be a random variable that represents the number of throws at which we throw 9.

- a) Create the probability table of random variable X (write values in the form of fractions).
 b) Calculate the expected value and the variance of random variable X .
 c) Calculate the probabilities $P(0 < X \leq 2), P(X > E(X))$.
 d) Write the distribution function prescription of random variable X .

$X \sim \text{bino}(3, \frac{1}{9})$ a) $\frac{x_i}{p_i} \parallel \begin{array}{c|c|c|c} 0 & 1 & 2 & 3 \\ \hline \frac{512}{729} & \frac{192}{729} & \frac{24}{729} & \frac{1}{729} \end{array}$; b) $E(X) = \frac{1}{3}, D(X) = \frac{8}{27}$; c) $P(0 < X \leq 2) = \frac{216}{729}, P(X > E(X)) = \frac{217}{729}$

3.39. There are 9 balls in the box, one third of which are white balls. Let X be a random variable that obtains the number of white balls between four randomly selected balls (without returning).

- a) Create the probability table of random variable X (write values in the form of fractions).
 b) Calculate the expected value of random variable X .
 c) Calculate the probabilities $P(0 < X \leq 2), P(X > E(X))$.
 d) Write the distribution function prescription of random variable X .

$X \sim \text{hyge}(9, 3, 4)$ a) $\frac{x_i}{p_i} \parallel \begin{array}{c|c|c|c} 0 & 1 & 2 & 3 \\ \hline \frac{15}{126} & \frac{60}{126} & \frac{45}{126} & \frac{6}{126} \end{array}$; b) $E(X) = \frac{4}{3}$; c) $P(0 < X \leq 2) = \frac{105}{126}, P(X > E(X)) = \frac{51}{126}$

3.40. Suppose we throw two dice at the same time. Let X be a random variable that represents the minimum of thrown values.

- a) Create the probability table of random variable X (write values in the form of fractions).

- b) Calculate the expected value and the variance of random variable X .
- c) Calculate the probabilities $P(X > 4)$, $X \leq E(X)$.
- d) Write the distribution function prescription of random variable X .

$$\text{a) } \begin{array}{c|c|c|c|c|c} x_i & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline p_i & \frac{11}{36} & \frac{9}{36} & \frac{7}{36} & \frac{5}{36} & \frac{3}{36} & \frac{1}{36} \end{array}; \text{ b) } E(X) = \frac{91}{36}, D(X) = \frac{2555}{1296}; \text{ c) } P(X > 4) = \frac{1}{9}, P(X \leq E(X)) = \frac{5}{9}$$

3.41. The target is a K circle and two annular rings M_1 and M_2 . Hit K means 10 points, hit M_1 means 5 points, and M_2 means 0 points. The probability of hit K is 0.5, the probability of hit M_1 is 0.3 and the probability of hit M_2 is 0.2. Let the random variable X be equal to the number of points scored for 2 independent shots.

- a) Create the probability table of random variable X (write values in the form of fractions).
- b) Calculate the expected value of random variable X .
- c) Calculate the probabilities $P(X \leq 10)$, $P(5 < X \leq 15)$.
- d) Write the distribution function prescription of random variable X .

$$\begin{array}{c|c|c|c|c} x_i & 0 & 5 & 10 & 15 & 20 \\ \hline p_i & 0,04 & 0,12 & 0,29 & 0,3 & 0,25 \end{array}; \text{ b) } E(X) = 13; \text{ c) } P(X \leq 10) = 0,45; P(5 < X \leq 15) = 0,59.$$

3.42. There are 15 students in the classroom, one third of whom study with honors. Let X be a random variable that obtains the number of honored students among three randomly selected students.

- a) Create the probability table of random variable X (write values in the form of fractions).
- b) Calculate the probability that at least one honored student was among the three randomly selected students.
- c) Calculate the expected value of random variable X .
- d) Write the distribution function prescription of random variable X .

$$X \sim \text{hyge}(15, 5, 3); \text{ a) } \begin{array}{c|c|c|c} x_i & 0 & 1 & 2 & 3 \\ \hline p_i & \frac{120}{455} & \frac{225}{455} & \frac{100}{455} & \frac{10}{455} \end{array}; \text{ b) } E(X) = 1; \text{ c) } P(0 < X \leq 2) = \frac{325}{455}, P(X > E(X)) = \frac{120}{455}$$

3.43. The probability of a correct answer for competition is 0.7. The President of the Commission asks questions. If the respondent answers incorrectly, he / she will not receive another question and will be excluded from the candidates. If he answers correctly, he gets another question, but together they ask him up to 4 questions.

- a) Create the probability table of random variable X .

- b) Calculate the expected value of random variable X .
- c) Calculate the probability that at least 3 questions were asked.
- d) Write the distribution function prescription of random variable X .

$$\begin{array}{c|c|c|c} x_i & 1 & 2 & 3 & 4 \\ \hline p_i & 0,3 & 0,21 & 0,147 & 0,343 \end{array}; \text{ b) } E(X) = 2,533; \text{ c) } P(X \geq 3) = 0,49.$$

3.44. The probability of making a standard car part is 0.8. From the series of parts produced, the controller gradually selects the parts and checks their quality. If the part is of good quality, the controller selects the next one, but checks a maximum of 4 parts. If the part to be checked does not comply with the standard, the check is stopped and the whole series is discarded. The random variable X takes the value of the number of parts to be checked.

- a) Create the probability table of random variable X .
- b) Calculate the probability that at least three parts were checked.
- c) Calculate the expected value and the variation of random variable X . Calculate the expected value of random variable X .

$$\begin{array}{c|c|c|c} x_i & 1 & 2 & 3 & 4 \\ \hline p_i & 0,2 & 0,16 & 0,128 & 0,512 \end{array}; \text{ b) } E(X) = 2,952; \text{ c) } P(X \geq 3) = 0,64.$$

3.45. The research showed that the monthly consumption of gasoline in the home has a normal distribution with a mean of 60 litres, and a standard deviation of 10 litres. Calculate the probability that

- a) randomly selected household consumes more than 70 litres,
 - b) randomly selected household consumption is in the range from 58 litres to 64 litres.
- [0.15866; 0.23468]

3.46. The height of five years old child has a normal distribution with a mean of 110 cm standard deviation of 6 cm. Calculate the probability that

- a) randomly selected child has a height less than 105 cm,
 - b) randomly selected child has a height greater than 120 cm,
 - c) randomly selected child has a height in the range from 110 cm to 140 cm.
- [0.20327; 0.04746; 0.5]

3.47. The tensile strength of randomly selected product has a normal distribution with a mean of 2.4 and a standard deviation of 0.6. Calculate the probability that

- a) randomly selected article has a tensile strength less than 4,

- b) randomly selected article has a tensile strength greater than 2.2,
- c) randomly selected article has a tensile strength in the range from 2.1 to 5.2.

[0.99609; 0.6293; 0.69146]

3.48. The height of an adult male has a normal distribution with a mean of 177 cm and a standard deviation of 3.4 cm. Calculate the probability that

- a) randomly selected male has a height less than 180 cm,
- b) randomly selected male has a height greater than 170 cm,
- c) randomly selected male has a height in the range from 170 cm to 192 cm.

[0.81057; 0.9803; 0.9803]

3.49. The elasticity of a particular product has a normal distribution with a mean of 3.5 and a standard deviation of 0.4. Calculate the probability that

- a) randomly selected product has a flexibility less than 3.7,
- b) randomly selected product has a flexibility greater than 3.1,
- c) randomly selected product has a flexibility in the range from 3.2 to 5.2. [0.69146; 0.84134; 0.77337]

3.50. In the tailor's workshop, they receive an average of 12 new orders in 5 working days. Calculate the probability that

- a) within 3 working days they receive 8 orders,
- b) within one working day they receive at least two orders. [a) 0,13372; b) 0,69156]

3.8 Results of Unsolved Tasks

3.1.

a) $k = 2/9$;

b)
$$F(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x^2/3 - 2x^3/27 & \text{for } x \in (0; 3), \\ 1 & \text{for } x > 3. \end{cases}$$

c) $E(X) = 1,5$, $D(X) = 0,45$, $\sigma(X) = 0,6708$;

d) $P(1 \leq X < 2) = 13/27$.

3.2.

a) $k = 3$;

b)
$$F(x) = \begin{cases} 0 & \text{for } x < 1, \\ 1 - 1/x^3 & \text{for } x \geq 1. \end{cases}$$

c) $E(X) = 1,5$ and $D(X) = 0,75$;

d) $P(\sqrt[3]{2} \leq X < E(X)) = 11/54$.

3.3.

a) $k = 2$;

b)
$$F(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ \sin 2x & \text{for } x \in (0, \pi/4), \\ 1 & \text{for } x > \pi/4. \end{cases}$$

c) $E(X) = \pi/4 - 1/2$ and $D(X) = 0,0354$;

d) $P(-3 < X < \pi/12) = 0,5$.

3.4.

a) $a = 1/3$, $b = -1$, $c = 1$;

b)
$$f(x) = \begin{cases} 1/3 & \text{for } x \in (3, 6), \\ 0 & \text{otherwise.} \end{cases}$$

c) $E(X) = 4,5$ and $D(X) = 0,75$;

d) $P(4 < X \leq 5) = 1/3$ and $P(-0,5 < X \leq 5) = 2/3$

3.5.

a) $a = 1/7$, $b = 5$;

b) $f(x) = \begin{cases} 1/7 & \text{for } x \in (-5, 2), \\ 0 & \text{otherwise.} \end{cases}$

c) $P(-2 < X \leq 2) = 4/7$, $P(-6 < X \leq 1) = 6/7$.

3.6.

a) $a = 1$, $b = 1$;

b) $f(x) = \begin{cases} 2x - x^3 & \text{for } x \in (0, \sqrt{2}), \\ 0 & \text{otherwise.} \end{cases}$

c) $E(X) = 8\sqrt{2}/15$, $D(X) = 22/225$.

3.7.

a) $a = 0$, $b = 1$, $c = 0$, $d = 1$;

b) $f(x) = \begin{cases} \cos x & \text{for } x \in \langle 0, \pi/2 \rangle, \\ 0 & \text{otherwise.} \end{cases}$

c) $E(X) = 0,5708$, $D(X) = 0,1416$;

d) $P(0 < X < \pi/4) = 0,7071$, $P(0 \leq X < \pi/2) = 1$.

3.8.

a) $a = 1$, $b = -1$, $d = 0$;

b) $f(x) = \begin{cases} e^{-x} & \text{for } x \in (0, \infty), \\ 0 & \text{otherwise.} \end{cases}$

c) $P(1 < X \leq 4) = 1/e - 1/e^4$, $P(X \geq 2) = 1/e^2$, $P(0 < X \leq 3) = 1 - 1/e^3$;

d) $E(X) = 1$ and $D(X) = 1$.

3.9.

- a) $a = 1/2$, $b = 1/\pi$;
- b) $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in (-\infty, \infty)$.
- c) $P(1/\sqrt{3} < X < 1) = 1/12$, $P(-1 < X < \sqrt{3}) = 7/12$;
- d) $E(X)$ and $D(X)$ do not exist.

3.10.

- a) discrete;
- b) continuous;
- c) discrete;
- d) continuous;
- e) continuous;
- f) continuous.

3.11. $X \sim \text{poiss}(3)$

- a) $P(X = 3) = \frac{3^3 \cdot e^{-3}}{3!} = 0,0504$;
- b) $P(X \geq 2) = 1 - \sum_{x=0}^1 \frac{3^x \cdot e^{-3}}{x!} = 0,8006$;
- c) $P(X \leq 4) = \sum_{x=0}^4 \frac{3^x \cdot e^{-3}}{x!} = 0,8153$.

3.12. $X \sim \text{bino}(10; 0,59)$, $P(X = 6) = \binom{10}{6} \cdot 0,59^6 \cdot 0,41^4 = 0,2506$ **3.13.** $X \sim \text{poiss}(10)$

- a) $P(X \geq 12) = 1 - \sum_{x=0}^{11} \frac{10^x \cdot e^{-10}}{x!} = 0,3032$;
- b) $P(X \leq 8) = \sum_{x=0}^8 \frac{10^x \cdot e^{-10}}{x!} = 0,3328$;
- c) $P(7 < X < 12) = \sum_{x=8}^{11} \frac{10^x \cdot e^{-10}}{x!} = 0,2377$.

3.14. $X \sim \text{bino}(10; 0,65)$

a) $P(X = 6) = \binom{10}{6} \cdot 0,65^6 \cdot 0,35^4 = 0,2377;$

b) $P(X < 3) = \sum_{x=0}^2 \binom{10}{x} \cdot 0,65^x \cdot 0,35^{10-x} = 0,0048;$

c) $P(5 < X < 9) = \sum_{x=6}^8 \binom{10}{x} \cdot 0,65^x \cdot 0,35^{10-x} = 0,6655.$

3.15. $X \sim \text{bino}(10; 0,7)$

a) $P(X = 8) = \binom{10}{8} \cdot 0,7^8 \cdot 0,3^2 = 0,2355;$

b) $P(X \leq 7) = \sum_{x=0}^7 \binom{10}{x} \cdot 0,7^x \cdot 0,3^{10-x} = 0,6172;$

c) $P(X \geq 8) = \sum_{x=8}^{10} \binom{10}{x} \cdot 0,7^x \cdot 0,3^{10-x} = 0,3828;$

d) $P(3 \leq X \leq 7) = \sum_{x=3}^7 \binom{10}{x} \cdot 0,7^x \cdot 0,3^{10-x} = 0,6156;$

e) $P(4 < X < 9) = \sum_{x=5}^8 \binom{10}{x} \cdot 0,7^x \cdot 0,3^{10-x} = 0,8034;$

f) $P(X < 5) = \sum_{x=0}^4 \binom{10}{x} \cdot 0,7^x \cdot 0,3^{10-x} = 0,0473;$

g) $P(X > 3) = \sum_{x=4}^{10} \binom{10}{x} \cdot 0,7^x \cdot 0,3^{10-x} = 0,9894.$

3.16. $X \sim \text{exp}(600)$

a) $P(X < 800) = 1 - e^{-800/600} = 0,7364;$

b) $P(X > 500) = e^{-500/600} = 0,2081;$

c) $P(600 < X < 700) = e^{-600/600} - e^{-700/600} = 0,0565.$

3.17. $X \sim \text{norm}(200; 15)$

a) $P(X < 195) = 1 - \Phi(0,333) = 0,36943;$

b) $P(X > 190) = \Phi(0,667) = 0,74961;$

c) $P(180 < X < 210) = \Phi(0,667) + \Phi(1,333) - 1 = 0,65635.$

3.18. $X \sim \text{norm}(18; 4)$, $P(X > 20 | X > 15) = \frac{1 - \Phi(0, 5)}{\Phi(0, 75)} = 0, 39895$.

3.19. $X \sim \text{norm}(30; 4)$

$P(X < t) = 0, 07 \Rightarrow t = 30 - 4 \cdot \Phi^{-1}(0, 93) = 24, 097$. The company would probably give warranty for 24 months.

3.20. $X \sim \text{poiss}(14/3)$

a) $P(X = 4) = \frac{(14/3)^4 \cdot e^{-14/3}}{4!} = 0, 1858$;

b) $P(X \geq 3) = 1 - \sum_{x=0}^2 \frac{(14/3)^x \cdot e^{-14/3}}{x!} = 0, 8443$;

c) $P(3 \leq X \leq 6) = \sum_{x=3}^6 \frac{(14/3)^x \cdot e^{-14/3}}{x!} = 0, 6534$.

3.21. $X \sim \text{poiss}(\lambda)$

a) $\lambda = 2, 5$; $P(X = 0) = e^{-2,5} = 0, 0821$;

b) $\lambda = 1, 5$; $P(X > 3) = 1 - \sum_{x=0}^3 \frac{1, 5^x \cdot e^{-1,5}}{x!} = 0, 0656$;

c) $\lambda = 5$; $P(X < 5) = \sum_{x=0}^4 \frac{5^x \cdot e^{-5}}{x!} = 0, 4405$.

3.22. $X \sim \text{exp}(2)$

a) $P(X \leq 3) = 1 - e^{-3/2} = 0, 7769$;

b) $P(X > 4) = e^{-4/2} = 0, 1353$;

c) $P(1, 5 < X < 3) = e^{-1,5/2} - e^{-3/2} = 0, 2492$;

d) $P(X < t) = 0, 7 \Rightarrow t = -2 \cdot \ln 0, 3 = 2, 4079$;

e) $P(X > t) = 0, 6 \Rightarrow t = -2 \cdot \ln 0, 6 = 1, 0217$.

3.23. $X \sim \text{bino}(10; p)$

a) $p = 0, 8$; $P(X = 10) = \binom{15}{10} \cdot 0, 8^{10} \cdot 0, 2^5 = 0, 9389$;

b) $p = 0, 25$; $P(7 \leq X \leq 8) = \binom{15}{7} \cdot 0, 25^7 \cdot 0, 75^8 + \binom{15}{8} \cdot 0, 25^8 \cdot 0, 75^7 = 0, 0524$.

3.24. $X \sim \exp(7, 5)$

- a) $P(X > 10) = e^{-10/7,5} = 0,2636$;
- b) $P(X < 8) = 1 - e^{-8/7,5} = 0,6558$;
- c) $P(X > t) = 0,6 \Rightarrow t = -7,5 \cdot \ln 0,6 = 3,8312$;
- d) $P(X < t) = 0,9 \Rightarrow t = -7,5 \cdot \ln 0,1 = 17,269$.

3.25. $X \sim \text{unif}(2; 20)$

- a) $P(10 < X < 15) = 5/18$;
- b) $P(X < 5 | X < 10) = \frac{P(X < 5)}{P(X < 10)} = \frac{3/18}{8/18} = 3/8$;
- c) $P(X < m) = 0,35 \Rightarrow m = 8,3$;
- d) $P(X > m) = 0,5 \Rightarrow m = 11$.

3.26. $X \sim \text{bino}(25; 0,45)$

- a) $E(X) = 11,25$
- b) $\mathcal{M}o(X) = 11, P(X = \mathcal{M}o(X)) = 0,1583$;
- c) $P(X \geq 10) = 0,7576$;
- d) $P(5 \leq X \leq 15) = 0,9537$.

3.27. $X \sim \text{bino}(5000; 0,001)$

- a) $P(5 < X < 10) = 0,3523$;
- b) $E(X) = 5, D(X) = 4,9950$ and $\sigma(X) = 2,2349$.

3.28. $X \sim \text{poiss}(6)$

- a) $P(X = 2) = 0,0446$;
- a) $P(X \leq 2) = 0,0620$;
- a) $P(X \geq 2) = 0,9826$;
- d) $P(X < t) = 0,99 \Rightarrow t = 12$.

3.29. $X \sim \text{hyge}(80, 12, 8)$

- a) $P(X = 5) = 0,0014$;

b) $P(X \geq 1) = 0,7450$;

c) $P(X < 5) = 0,9999$;

d) $\mathcal{M}o(X) = 1$;

e) $E(X) = 1,2$.

3.30. $X \sim \text{unif}(0, b)$

a) $P(X \geq 4) = 0,6 \Rightarrow 4/b = 0,4 \Rightarrow b = 10$

b) $E(X) = 5$ and $\sigma(X) = 2,8868$;

c)
$$F(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ \frac{x}{10} & \text{for } x \in (0; 10), \\ 1 & \text{for } x \geq 10. \end{cases}$$

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List of Tables

Lexicon – Vocabulary

English – Slovak

A

- absolute frequency
- absolute frequency distribution
- accept the hypothesis
- acceptance
- acceptance region
- ace
- alternate hypothesis
- at random
- atomic outcome
- attribute
- absolútna početnosť
- rozdelenie absolútnej početnosti
- prijať hypotézu
- prijatie
- oblasť (obor) prijatia
- eso
- alternatívna hypotéza
- náhodne
- elementárny jav
- (štatistický) znak

B

- bar chart
- base of vector space
- Bayes' formula
- Bayes' theorem
- Bernoulli trials
- the best unbiased estimator
- binomial coefficient
- binomial distribution
- binomial number
- binomial n k , $\binom{n}{k}$
- stĺpcový (tyčkový) diagram (graf)
- báza vektorového priestoru
- Bayesov vzorec
- Bernoulliho schéma
- nejlepší nestranný odhad
- binomický koeficient (číslo)
- binomické rozdelenie
- binomické číslo
- n nad k , $\binom{n}{k}$

C

- Cauchy[']s] distribution
- central limit theorem
- central moment
- certain (sure) event
- Cauchyovo rozdelenie
- centrálna limitná veta
- centrálny moment
- istý jav

- chi-squared distribution
- choose, n choose k , $\binom{n}{k}$
- class
- class boundary (upper, lower)
- class frequency
- class interval
- class mark
- clubs
- combination theory
- combination of n objects taken k at a time
- combinatorics
- complementary event (A^C)
- conditional probability
- the conditional probability of event A given that event B has occurred
- confidence interval
- consistent estimator
- continuous random variable
- critical region
- critical value
- cross (pile), heads (tails)
- cumulative frequency
- cumulative relative frequency distribution
- cumulative relative frequency
- chí-kvadrát rozdelenie
- n nad k , $\binom{n}{k}$
- trieda
- hranica triedy (horná, dolná)
- triedna početnosť
- triedny interval
- triedny znak (zástupca triedy)
- žalud, trojlístok
- kombinatorika
- kombinácie k -tej triedy z n prvkov
- kombinatorika
- opačný jav (A^C)
- podmienená pravdepodobnosť
- podmienená pravdepodobnosť javu A za predpokladu, že nastal jav B
- interval spoľahlivosti
- konzistentný odhad
- spojitá náhodná premenná
- kritický obor (oblasť, interval)
- kritická hodnota
- panna (lev), hlava (orol)
- kumulatívna početnosť
- empirická distribučná funkcia
- kumulatívna relatívna početnosť

D

- deal (cards)
- decile
- decision-making situation
- degree of freedom
- descriptive statistics
- diamonds
- die (pl. dice)
- dice-box (dice-cup)
- discrete random variable
- distribute (cards)
- distribution
- distribution function
- rozdať (karty)
- decil
- rozhodovacia situácia
- stupeň voľnosti
- popisná štatistika
- guľa, piky
- kocka (množ. kocky)
- pohár na miešanie kociek
- diskrétna náhodná premenná
- rozdať (karty)
- rozdelenie
- distribučná funkcia

E

- elementary event
- elementary outcome
- empirical probability distribution
- equally likely
- establish the critical value
- estimator
- event
- excess (kurtosis)
- expected value
- expectation
- exponential (probability) distribution
- elementárny jav
- elementárny jav
- empirické rozdelenie pravdepodobnosti
- rovnako pravdepodobný
- určiť kritickú hodnotu
- odhad (štatistika)
- jav
- špicatosť (ostroť)
- stredná hodnota
- stredná hodnota
- exponenciálne rozdelenie (pravdepodobnosti)

F

- face (of a coin, card)
- face (side)
- failure probability
- fair die
- favourable outcome
- Fisher[’s] F -distribution
- flip a coin
- frequency polygon
- líc (mince, kasty)
- stena kocky
- pravdepodobnosť neúspechu
- regulárna kocka
- priaznivý výsledok
- Fisherovo-Snedecorovo F -rozdelenie
- hodiť mincu
- polygón početnosti

G

- game of chance
- Gaussian (probability) distribution
- geometric probability
- goodness-of-fit test
- group
- grouped data
- hazardná hra
- normálne (Gaussovo) rozdelenie (pravdepodobnosti)
- geometrická pravdepodobnosť
- test dobrej zhody
- zlúčiť
- usporiadané (utriedené) údaje (data)

H

- happen
- hand (of cards)
- heads (tails), cross (pile)
- hearts
- histogram
- hypergeometric distribution
- hypothesis testing
- stať sa, uskutočniť sa, prihodiť sa
- list (kariet)
- hlava (orol), panna (lev)
- červeň, srdce
- histogram
- hypergeometrické rozdelenie
- testovanie hypotéz

I

- impossible event
- inclusion-exclusion identity
- independent events
- inferential statistics
- interquartile range (*IQR*)
- intersection of events
- interval estimate (estimation)
- nemožný jav
- princíp inklúzie a exklúzie
- nezávislé javy
- štatistická indukcia
- medzikvantilové rozpätie (*IQR*)
- prienik (súčin) javov
- intervalový odhad

J

- jack (knave)
- nižník

K

- k -th moment (moment of order k)
- king
- Kolmogorov-Smirnov test
- kurtosis (excess)
- k -tý moment (moment k -teho rádu)
- kráľ
- Kolmogorov-Smirnov test
- špicatosť (ostroť)

L

- leptokurtic
- law of large numbers
- law of total probability
- level of confidence
- level of significance
- likelihood
- local minimum
- lower quartile
- špicatý
- zákon veľkých čísel
- zákon o úplnej pravdepodobnosti
- hladina spoľahlivosti
- hladina významnosti
- šanca, pravdepodobnosť
- lokálne minimum
- dolný kvartil

M

- mathematical statistics
- mean
- measure of location
- measure of dispersion
- measure of skewness
- measure of kurtosis
- median
- method of maximum likelihood
- mesokurtic
- mode
- mutually disjoint events
- matematická štatistika
- stredná hodnota (priemer)
- charakteristika (miera) polohy
- charakteristika (miera) variability
- charakteristika (miera) šikmosti
- charakteristika (miera) špicatosti (ostrosti)
- medián
- metóda maximálnej vierohodnosti
- normálne špicatý
- modus
- vzájomne nezlučiteľné javy

- mutually exclusive events

- navzájom nezlučiteľné javy

N

- natural number

- prirodzené číslo

- negatively skewed (skewed to the right)

- záporne zošikmený (zošikmený vpravo)

- normal (Gaussian) distribution

- normálne rozdelenie

- normed random variable

- normovaná náhodná premenná

- null hypothesis

- nulová hypotéza

O

- object is drawn at random

- objekt je náhodne vytiahnutý

- object is pulled at random

- objekt je náhodne vytiahnutý

- observation

- pozorovanie

- observed value

- pozorovaná hodnota

- on a single draw

- jedným ťahom

- occur

- stať sa, uskutočniť sa, prihodiť sa

- occurrence

- výskyt

- on a single pull

- jedným ťahom

- one-sided confidence interval

- jednostranný interval spoľahlivosti

- one-sided test

- jednostranný test

- order statistic

- usporiadaná (poradová) štatistika

- outcome

- výsledok pokusu

P

- p -th quantile (quantile of order p)

- $100p$ -percentný kvantil

- the p percent confidence interval

- p percentný interval spoľahlivosti

- the p percent level of significance

- p percentná hladina významnosti

- pack of cards (deck)

- balíček kariet

- pairwise exclusive events

- po dvoch nezlučiteľné javy

- Pearson[']s χ^2 -distribution

- Pearsonovo χ^2 -rozdelenie

- percentile

- percentil

- permutation of n objects (things)

- permutácia n prvkov

- permutation of n objects

- variácie k -tej triedy z n prvkov

taken k at a time

- pie chart

- kruhový (koláčový) diagram (graf)

- platykurtic

- plochý

- point estimate (estimation)

- bodový odhad

- Poisson distribution

- Poissonovo rozdelenie

- population

- (základný) štatistický súbor, populácia

- positively skewed (skewed to the left)

- pozitívne zošikmený (zošikmený vľavo)

- power of the test

- sila testu

- probability

- pravdepodobnosť

- probability density function
- probability function
- probability mass function
- the probability of A given B ,
($P(A|B)$)
- probability space
- probability theory
- the probability that A occurs,
given that B occurred
- possible outcome
- pull out
- pulling
- pulling out
- hustota pravdepodobnosti
- pravdepodobnostná funkcia
- pravdepodobnostná funkcia
- pravdepodobnosť javu A ,
ak nastal jav B , ($P(A|B)$)
- pravdepodobnostný priestor
- teória pravdepodobnosti
- podmienená pravdepodobnosť, že nastane
jav A za predpokladu, že jav B nastal
- možný výsledok
- vytiahnuť
- ťahanie
- vyťahovanie

Q

- quantile
- quartile (lower, upper)
- queen
- kvantil
- kvartil (dolný, horný)
- vyšník, dáma, kráľovná

R

- random
- random experiment
- random sample
- random sampling
- random sample of n observations
- random variable
- random variate
- range
- region of acceptance
- region of rejection
- reject the hypothesis
- rejection
- rejection probability
- rejection region
- relative frequency
- relative frequency distribution
- roll a die
- roll a number
- roll a number on the first roll
- roll a number on the second roll
- roll a six
- roll a six on the first roll
- náhodný
- náhodný pokus
- náhodný výber, vzorka
- náhodný výber (vyberanie, selekcia)
- náhodný výber n pozorovaní
- náhodná premenná
- náhodná veličina
- variačné rozpätie
- oblasť (obor) prijatia
- oblasť (obor) zamietnutia
- zamietnuť hypotézu
- zamietnutie
- pravdepodobnosť neúspechu
- oblasť (obor) zamietnutia
- relatívna početnosť
- rozdelenie relatívnej početnosti
- hodiť kocku
- hodiť číslo
- hodiť číslo prvým hodom
- hodiť číslo druhým hodom
- hodiť šestku
- hodiť šestku prvým hodom

- roll a six on the second roll

- hodit šestku v druhom hode

S

- sample function

- výberová funkcia (štatistika)

- sample mean

- výberový priemer

- sample size

- veľkosť náhodného výberu

- sample space

- základný priestor,
priestor elementárnych javov

- sample statistic

- výberová štatistika (funkcia náhodného výberu)

- sample variance

- výberový rozptyl

- sampling distribution

- výberové rozdelenie

- shake dice

- zamiešať kocky

- shuffle

- zamiešať (pomiešať)

- side (face)

- stena kocky

- significance level

- hladina významnosti

- simple event

- elementárny jav

- simple random sample

- prostý náhodný výber

- single-sample test

- jednovýberový test

- six-sided die

- šesťstenná kocka

- size of a sample

- rozsah náhodného výberu

- skewness

- šikmosť

- spades

- zeleň, listy

- staircase (step) function

- schodkovitá (stupňovitá) funkcia

- standard deviation

- smerodajná odchýlka

- standardized normal distribution

- štandardizované (normované) normálne rozdelenie

- standardized random variable

- štandardizovaná náhodná premenná

- statistic

- štatistika (funkcia náhodného výberu)

- statistical

- štatistický

- statistical decision

- štatistické rozhodovanie

- statistical hypothesis (pl. hypotheses)

- štatistická hypotéza (mnž. hypotézy)

- statistical inference

- štatistický záver

- statistically significant

- štatisticky významný

- statistician

- štatistik

- Student[']s t -distribution

- Studentovo t -rozdelenie

- success probability

- pravdepodobnosť úspechu

- suit (of cards)

- farba (kariet)

- survey

- skúmanie (prieskum)

- symmetrical

- symetrický

T

- test for two samples

- dvojjvýberový test

- test statistic

- testovacia štatistika (charakteristika), kritérium

-
- theory of probability
 - throw a die
 - to be p percent confident
 - toss a coin
 - total (composite) probability
 - total probability formula
 - total probability theorem
 - trial
 - two separate tosses
 - two-sided confidence interval
 - two-sided test
 - Type I error
 - Type II error
 - teória pravdepodobnosti
 - hodit kockou
 - spoľahnúť sa s p percentnou istotou
 - hodit mincu
 - úplná pravdepodobnosť
 - vzorec pre úplnú pravdepodobnosť
 - veta o úplnej pravdepodobnosti
 - jedna realizácia náhodného pokusu
 - dva samostatné hody
 - obojstranný (dvojstranný) interval spoľahlivosti
 - dvojstranný test
 - chyba prvého druhu
 - chyba druhého druhu

U

- unbiased estimator
- uniform distribution
- union of events
- upper quartile
- urn
- nestranný odhad
- rovnomerné rozdelenie
- zjednotenie (súčet) javov
- horný kvartil
- urna

V

- variance
- rozptyl

W

- Weibull[']s] distribution
- with repetition
- with replacement
- without repetition
- without replacement
- Weibullovo rozdelenie
- s opakovaním
- s vracaním
- bez opakovania
- bez vracania

Z

- zero hypothesis
- nulová hypotéza

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Autor: © RNDr. Štefan BEREŽNÝ, PhD., 2019
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Recenzovali: RNDr. Kristína Budajová, PhD.
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