POSETS AND LATICES

BOOLEAN ALGEBRA AND BOOLEAN FUNCTIONS

PARTIALLY ORDERED SETS

Definition

A partially ordered set is a pair $(A, \mathcal{R})$ where A is a set and $\mathcal{R}$ is a relation on A that satisfies the following:

- *$\mathcal{R}$ is reflexive: $\forall x \in A : x\mathcal{R}x$*
- *$\mathcal{R}$ is antisymmetric: $\forall x, y \in A :$ if $x\mathcal{R}y$ and $y\mathcal{R}x$, then $x = y$*
- *$\mathcal{R}$ is transitive: $\forall x, y, z \in A :$ if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$*

The relation $\mathcal{R}$ is called partial order relation. The term poset is an abbreviation for partially ordered set.

Example

- Relation $\mid$ (divides) on the naturals numbers $\mathbb{N}$.
- Relation $\leq$ on the integers $\mathbb{R}$.
- Relation $\subseteq$ on the power set $\mathcal{P}(\mathcal{A})$, for some set $A \neq \emptyset$.

PARTIALLY ORDERED SETS

Each element of A is represented by a dot in the plane. If $x\mathcal{R}y$ in the poset, then we draw $x's$ dot below $y's$ dot and draw a line (or curve) from x to y.

We do not need to draw a curve from a dot to itself. We know that partial ordered relations are reflexive, we do not need the diagram to remind us of this fact.

Because partial ordered relations are transitive, we can infer $x\mathcal{R}z$ from the diagram. We can read this in the diagram by following an upward path from x through y to z. By not drawing a curve from x to z, the diagram is less cluttered and easier to read.

These diagrams of posets are known as Hasse diagrams.

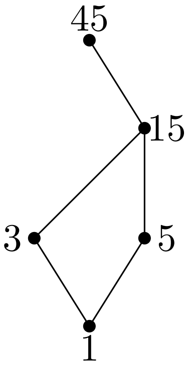
Example

Draw the Hasse diagram of the poset $(A, \leq)$, $A = \{1, 2, 4, 40\}$.

PARTIALLY ORDERED SETS

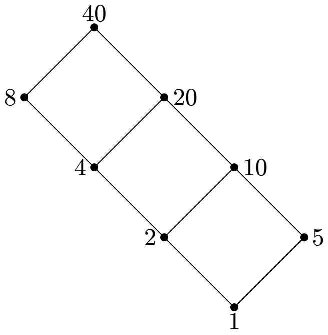
Example

Draw the Hasse diagram of the poset $(A, |)$, $A = \{1, 3, 5, 15, 45\}$.



Example

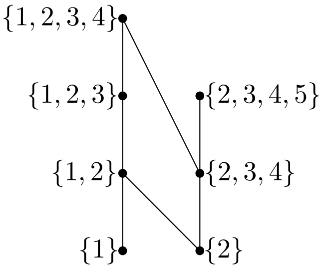
Draw the Hasse diagram of the poset $(A, \mid)$, $A = \{1, 2, 4, 5, 8, 10, 20, 40\}$.



PARTIALLY ORDERED SETS

Example

Draw the Hasse diagram for the poset $(A, \subseteq)$, $A = \{\{1\}, \{2\}, \{1,2\}, \{2,3,4\}, \{1,2,3\}, \{2,3,4,5\}, \{1,2,3,4\}\}$.



PARTIALLY ORDERED SETS

Definition

Let $(A, \mathcal{R})$ be a poset and let $x, y \in A$. We call the elements x and y comparable provided $x\mathcal{R}y$ or $y\mathcal{R}x$. We call the elements x and y incomparable if neither $x\mathcal{R}y$ nor $y\mathcal{R}x$.

Definition

Let $(A, \mathcal{R})$ be a poset.
An element $x \in A$ is called maximum, if for all $a \in A$ we have $a\mathcal{R}x$.
We call x minimum if for all $b \in A$, we have $x\mathcal{R}b$.

$x \in A$ is maximum, if all other elements of the poset are below x, and x is minimum if all other elements of the poset are above x.

PARTIALLY ORDERED SETS

Definition

Let $(A, \mathcal{R})$ be a poset.
An element $x \in A$ is called maximal, if there is no $a \in A$, $a \neq x$ with $x\mathcal{R}a$.
An element x is called minimal if there is no $a \in A$, $a \neq x$, with $a\mathcal{R}x$.

$x \in A$ is maximal, if there is no element strictly above x, and x is minimal if there is no element strictly below x.

Term	Meaning
maximum	all other elements are below
maximal	no other element is above
minimum	all other elements are above
minimal	no other element is below

Definition

Let $(A, \mathcal{R})$ be a poset and let $a, b \in A$. We say that $x \in A$ is a lower bound for a and b provided $x\mathcal{R}a$ and $x\mathcal{R}b$.
Similarly, we say that $x \in A$ is an upper bound for a and b provided $a\mathcal{R}x$ and $b\mathcal{R}x$.

Definition

Let $(A, \mathcal{R})$ be a poset and let $a, b \in A$.
We say that $x \in A$ is a greatest lower bound for a and b provided

(a) *x is a lower bound for a and b and*
(b) *if y is a lower bound for a and b then $y\mathcal{R}x$*

Similarly, we say that $x \in A$ is a least upper bound for a and b provided

(a) *x is an upper bound for a and b and*
(b) *if y is an upper bound for a and b then $x\mathcal{R}y$*

LATTICES

Definition

Let $(A, \mathcal{R})$ be a poset and let $a, b \in A$.
If a and b have a greatest lower bound, it is called the **meet** *of a and b, and it is denoted $a \wedge b$.*
If a and b have a least upper bound, it is called the **join** *of a and b, and it is denoted $a \vee b$.*

Definition

Let $(A, \mathcal{R})$ be a poset. We call $(A, \mathcal{R})$ a **lattice** *provided, for all elements a and b of A, $a \wedge b$ and $a \vee b$ are defined.*

Theorem

Let $(A, \mathcal{R})$ be a latice. For all $x, y, z \in A$, the following hold:

1. $x \wedge x = x, \qquad x \vee x = x$ *(Idempotent laws)*
2. $x \wedge y = y \wedge x, \qquad x \vee y = y \vee x$ *(Commutative laws)*
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \qquad x \vee (y \vee z) = (x \vee y) \vee z$ *(Associative laws)*
4. $x \wedge (y \vee z) = x, \qquad x \vee (y \wedge z) = x$ *(Absorption laws)*

LATTICES

Let A be a finite set. In every lattice $(A, \mathcal{R})$ there is maximum element which we denote I, and there is minimum element which we denote O.

Definition

Let $(A, \mathcal{R})$ be a lattice and $x \in A$. The element $x' \in A$ is a called complement to the element x if $x \vee x' = I$ and $x \wedge x' = O$. A lattice $(A, \mathcal{R})$ is called complementary if there is at least one complement to every element $x \in A$.

Definition

Let $(A, \mathcal{R})$ be a lattice. We call $(A, \mathcal{R})$ distributive provided, for all $x, y, z \in A$, the following hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Definition

A complementary and distributive lattice is called Boolean lattice.

Definition

*Let $\wedge$ and $\vee$ be two binary operations and let $'$ be unary operation. For $B \neq \emptyset$, the algebraic system $(B, \wedge, \vee, ', O, I)$ is called **Boolean algebra** provided for all $x, y, z \in B$ the following identities hold:*

1. $x \wedge y = y \wedge x, \qquad x \vee y = y \vee x$ *(Commutative laws),*
2. $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \qquad x \vee (y \vee z) = (x \vee y) \vee z$ *(Associative laws),*
3. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ *(Distributive laws)*
4. $x \wedge x = x, \qquad x \vee x = x$ *(Identity laws),*
5. $x \wedge x' = O, \qquad x \vee x' = I$ *(Complement laws),*

Theorem

Let $(B, \wedge, \vee, ', O, I)$ *be a boolean algebra. For all* $x, y \in B$ *the following hold:*

1. $x \wedge x = x, \qquad x \vee x = x$ *(Idempotent laws)*
2. $x \wedge O = O, \qquad x \vee I = I$ *(Dominance laws)*
3. $((x'))' = x$ *(Law of the double complement)*
4. $(x \wedge y)' = x' \vee y', \qquad (x \vee y)' = x' \wedge y'$ *(DeMorgan laws)*
5. $x \wedge (y \vee z) = x, \qquad x \vee (y \wedge z) = x$ *(Absorption laws)*

BOOLEAN ALGEBRA

Let us have the set $D = \{0, 1\}$. Let for the elements of D are defined two binary operations and one unary operations as shown in the tables:

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

	$'$
0	1
1	0

In this case the minimum element is O and the maximum element is 1.
For the operations join $\vee$ and meet $\wedge$. in the Boolean algebra are usually used terms Boolean sum and Boolean product, respectively.
The complement, Boolean sum and Boolean product correspond to the logical operators $'$ (negation), $\vee$ (or) and $\wedge$ (and), respectively, where 0 corresponds to false and 1 corresponds to true.

BOOLEAN FUNCTIONS

Definition

Let $D = \{0, 1\}$. The variable x is called Boolean variable if it assumes values only from D. A function from $D^n = \{(x_1, x_2, x_3, \dots x_n), x_i \in D, 1 \leq i \leq n\}$ to D is called a Boolean function of degree n.

Boolean functions can be represented using expressions made up from the variables $x_1, x_2, x_3, \dots x_n$ are defined recursively as follows

- $0, 1, x_1, x_2, x_3, \dots x_n$ are Boolean expressions
- if E_1 and E_2 are Boolean expressions, then E_1', $E_1 \wedge E_2$, and $E_1 \vee E_2$ are Boolean expressions

Each Boolean expression represents a Boolean function. The value of this function are obtained by substitututing 0 and 1 for the variables in the expression.

BOOLEAN FUNCTIONS

A Boolean function of degree two is a function from a set with four elements, namely, pairs $(0,0),(0,1),(1,0),(1,1)$. Hence, there are 16 such a functions.

(x_1, x_2)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
(0,0)	0	0	1	1	0	0	0	1	0	1	1	1	0	1	0	1
(0,1)	0	1	1	0	0	0	1	0	1	0	1	1	0	1	1	0
(1,0)	1	0	0	1	0	1	0	0	1	1	0	1	0	1	1	0
(1,1)	1	1	0	0	1	0	0	0	1	1	1	0	0	1	0	1

Theorem

There exist 2^{2^n} different Boolean functions of degree n.

Problem:

Given the values of a Boolean function, how can a Boolean expression that represents this function be found?

This problem will be solved by showing that any Boolean function may be represented by a *Boolean sum of Boolean products* and also by a *Boolean product of Boolean sums* of the variables and their complements.

BOOLEAN FUNCTIONS

Definition

A minterm of a Boolean variables $x_1, x_2, \ldots x_n$ is a Boolean product $x_1^\star \wedge x_2^\star \wedge \cdots \wedge x_n^\star$ where $x_i^\star = x_i$ or $x_i^\star = x_i'$.
A maxterm of a Boolean variables $x_1, x_2, \ldots x_n$ is a Boolean sum $x_1^\star \vee x_2^\star \vee \cdots \vee x_n^\star$ where $x_i^\star = x_i$ or $x_i^\star = x_i'$.

A minterm has the value 1 for one and only one combination of values of its variables.
The minterm $x_1^\star \wedge x_2^\star \wedge \cdots \wedge x_n^\star$ is 1 if and only if each $x_i^\star$ is 1, and this occurs if and only if $x_i = 1$ when $x_i^\star = x_i$ and $x_i = 0$ when $x_i^\star = x_i'$.
A maxterm has the value 0 for one and only one combination of values of its variables.
The maxterm $x_1^\star \vee x_2^\star \vee \cdots \vee x_n^\star$ is 0 if and only if each $x_i^\star$ is 0, and this occurs if and only if $x_i = 0$ when $x_i^\star = x_i$ and $x_i = 1$ when $x_i^\star = x_i'$.

BOOLEAN FUNCTIONS

By taking Boolean sums of distinct minterms we can build up a Boolean expression with a specified set of values.
A Boolean sum of minterms has the value 1 when exactly one of the minterms in the sum has the value 1.
It has the value 0 for all other combinations of values of the variables.
Given a Boolean function, a Boolean sum of minterms can be formed that has the value 1 when this Boolean function has the value 1, and has the value 0 when the function has the value 0. The minterms in this Boolean sum correspond to those combinations of values for which the function has value 1. The sum of minterms that represents the function is called the sum–of–product expansion or the disjunctive normal form of the Boolean function.

BOOLEAN FUNCTIONS

It is also possible to find a Boolean expression that represents a boolean function by taking a Boolean product of Boolean sums.
A Boolean product of maxterms has the value 0 when exactly one of the maxterms in the product has the value 0.
It has the value 1 for all other combinations of values of the variables.
Given a Boolean function, a Boolean product of maxterms can be formed that has the value 0 when this Boolean function has the value 0, and has the value 1 when the function has the value 1. The maxterms in this Boolean product correspond to those combinations of values for which the function has value 0. The product of maxterms that represents the function is called the product–of–sum expansion or the conjunctive normal form of the Boolean function.

BOOLEAN FUNCTIONS

Example

x	y	z	$f(x,y,z)$	minterms	maxterms
0	0	0	1	$x' \wedge y' \wedge z'$	
0	0	1	0		$x \vee y \vee z'$
0	1	0	1	$x' \wedge y \wedge z'$	
0	1	1	1	$x' \wedge y \wedge z$	
1	0	0	0		$x' \vee y \vee z$
1	0	1	0		$x' \vee y \vee z'$
1	1	0	0		$x' \vee y' \vee z$
1	1	1	1	$x \wedge y \wedge z$	

Disjunctive normal form (sum–of–product expansion) of the Boolean function $f(x,y,z)$ is $(x' \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y \wedge z)$

Conjunctive normal form (product–of–sum expansion) of the Boolean function $f(x,y,z)$ is $(x \vee y \vee z') \wedge (x' \vee y \vee z) \wedge (x' \vee y \vee z') \wedge (x' \vee y' \vee z)$

BOOLEAN FUNCTIONS

Example

Determine conjunctive normal form of the Boolean function $f(x, y, z)$ whose disjunctive normal form is $(x \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x \wedge y' \wedge z)$.

Solution: The Boolean function $f(x, y, z)$ has value 1 only in arguments $(1, 0, 0), (0, 0, 1)$, and $(1, 0, 1)$, and has value 0 in other arguments.

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Conjunctive normal form of the Boolean function $f(x, y, z)$ is
$(x \vee y \vee z) \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z) \wedge (x' \vee y' \vee z')$