PROPOSITIONAL LOGIC

PROPOSITIONAL LOGIC

By a statement we will understand something which is said about the world, and something which has a true value. We can say that a statement is any sentence for which we can decide whether it is true or not.

The following sentences are statements.
„It rains."
„Today is Thursday."
„Two and three is nine."

The following sentences are not statements.
„Does it rain?"
„It should rain!"

PROPOSITIONAL LOGIC

Elementary statements – the structure of the statements is of no importance. We denote it by propositional variables $x, y, z, \ldots$

More complicated statements – are built from elementary statements using the logical connectives:

- **negation**, „it is not the case that . . . “, we denote it by $\;'$ or $\neg$ or $\overline{}$
- **conjunction**, „. . . and . . . “, we denote it by $\wedge$
- **disjunction**, „. . . or . . . “, we denote it by $\vee$
- **implication**, „if . . . , then . . . “, we denote it by $\Rightarrow$
- **equivalence**, „. . . if and only if . . . “, we denote it by $\Leftrightarrow$

PROPOSITIONAL LOGIC

Consider two elementary statements: „It rains." and „The sun is shining."

- The negation of the first statement is the statement
  „It is not the case that it rains." or „It does not rain."
- The conjunction of the two statements is the statement
  „It rains and the sun is shining."
- The disjunction of the two statements is the statement
  „It rains or the sun is shining."
- The implication of the first and second statement is the statement
  „If it rains, then the sun is shining."
- The equivalence of them is the statement
  „ It rains if and only if the sun is shining."

PROPOSITIONAL LOGIC

Definition

Given a non-empty set A of elementary statements (propositional variables). A finite sequence of elements of the set A, of logical connectives and parentheses is called a propositional formula*, if it is formed by the following rules:*

1. *Every propositional variable $x \in A$ is a propositional formula.*
2. *If α and β are propositional formulas, then so are $(\neg\alpha)$, and $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \Rightarrow \beta)$, $(\alpha \Leftrightarrow \beta)$.*
3. *Only sequences that were formed by using finitely many applications of rules 1 and 2, are propositional formulas.*

We will use two rules which make the notation more simple.
1. We omit the outward parentheses. We write $(\alpha \wedge \beta)$ instead of $(\alpha \wedge \beta)$.
2. We assume, that the unary connective $\neg$ „is stronger than“ each of the binary ones. Hence, we will write $\neg\beta \wedge \alpha$ instead of $(\neg\beta) \wedge \alpha$.

TRUTH VALUATIONS OF FORMULAS

If we know that some of our formulas are true or false, we can determine the truth value of a formula that results from these formulas by means of some logical connectives. We state following rules:

- $\neg\alpha$ is true if and only if α is false,
- $\alpha \land \beta$ is true if and only if α and β are both true,
- $\alpha \lor \beta$ is false if and only if α and β are both false,
- $\alpha \Rightarrow \beta$ is false if and only if α is true and β is false,
- $\alpha \Leftrightarrow \beta$ is true if and only if either both α and β are true or both are false.

We will use the following coding: „true“ is 1 and „false“ is 0.

TRUTH VALUATIONS OF FORMULAS

The properties that any truth valuation must have, can also be expressed in terms of the truth tables of the logical connectives. There are:

α	$\neg\alpha$
0	1
1	0

α	β	$\alpha \wedge \beta$	$\alpha \vee \beta$	$\alpha \Rightarrow \beta$	$\alpha \Leftrightarrow \beta$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

TRUTH VALUATIONS OF FORMULAS

Definition

A formula is called a tautology provided it is true for all truth valuations, it is called a contradiction provided it is false for all truth valuations. A formula is satisfiable provided there is at least one truth valuation for which the formula is true.

Definition

A set of formulas $\mathcal{M}$ is called a satisfiable provided there exists a truth valuation for which every formula from $\mathcal{M}$ is true. Otherwise, we say that the set is unsatisfiable.

TRUTH VALUATIONS OF FORMULAS

Example

Decide whether the following formulas

- $\alpha : (x \Rightarrow (y \wedge \neg y)) \Rightarrow \neg x$
- $\beta : (\neg x \vee y) \Leftrightarrow (y \Rightarrow x)$
- $\gamma : \neg((x \Rightarrow \neg x) \Leftrightarrow \neg x)$

are tautologies, or satisfiable formulas which are not tautologies, or contradictions.

Solution: We write the truth tables of all of these three formulas.

x	y	$y \wedge \neg y$	$x \Rightarrow (y \wedge \neg y)$	α	$\neg x \vee y$	$y \Rightarrow x$	β
0	0	0	1	1	1	1	1
0	1	0	1	1	1	0	0
1	0	0	0	1	0	1	0
1	1	0	0	1	1	1	1

x	$x \Rightarrow \neg x$	$(x \Rightarrow \neg x) \Leftrightarrow \neg x$	γ
0	1	1	0
1	0	1	0

α is a tautology, β is satisfiable formula but not a tautology, γ is a contradiction.

Example

Decide whether a set of formulas $\{y \Rightarrow x,\ y \vee \neg x\}$ is satisfiable or not.

Solution: We write the truth tables of all of these formulas.

x	y	$y \Rightarrow x$	$y \vee \neg x$
0	0	1	1
0	1	0	1
1	0	1	0
1	1	1	1

$\{y \Rightarrow x,\ y \vee \neg x\}$ is satisfiable.

SEMANTICAL CONSEQUENCE

Definition

We say that a formula α is a semantical consequence of a set of formulas $\mathcal{M}$ (or that α semantically follows from the set $\mathcal{M}$), provided α is true for every truth valuation for which every formula from $\mathcal{M}$ is true.

The fact that formula α is a semantical consequence of a set of formulas $\mathcal{M}$ we denote $\mathcal{M} \models \alpha$

Theorem

For a set of formulas $\mathcal{M}$ and a formula α we have

$$\mathcal{M} \models \alpha \text{ if and only if } \mathcal{M} \cup \{\neg\alpha\} \text{ is unsatisfiable.}$$

SEMANTICAL CONSEQUENCE

Example

Decide whether the following semantical consequences are valid or not.

a) $\overline{x \wedge y} \models z \Rightarrow y$

b) $\{x \Leftrightarrow y,\ \overline{x}\} \models z \Rightarrow \overline{y}$

c) $\{x \vee \overline{y},\ \overline{y} \wedge (z \vee x)\} \models x \Rightarrow y$

d) $\{p \vee q,\ q \Rightarrow (r \wedge \overline{p})\} \models q \Leftrightarrow r.$

Solution:

a) $\overline{x \wedge y} \models z \Rightarrow y$ is false.

x	y	z	$\overline{x \wedge y}$	$z \Rightarrow y$
0	0	0	1	1
0	0	1	1	0
0	1	0	1	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	0
1	1	0	0	1
1	1	1	0	1

SEMANTICAL CONSEQUENCE

b) $\{x \Leftrightarrow y,\ \overline{x}\} \models z \Rightarrow \overline{y}$ is true.

x	y	z	$x \Leftrightarrow y$	$\overline{x}$	$z \Rightarrow \overline{y}$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	0	1	1
0	1	1	0	1	0
1	0	0	0	0	1
1	0	1	0	0	1
1	1	0	1	0	1
1	1	1	1	0	0

SEMANTICAL CONSEQUENCE

c) $\{x \vee \overline{y},\ \overline{y} \wedge (z \vee x)\} \models x \Rightarrow y$ is false.

x	y	z	$x \vee \overline{y}$	$z \vee x$	$\overline{y} \wedge (z \vee x)$	$x \Rightarrow y$
0	0	0	1	0	0	1
0	0	1	1	1	1	1
0	1	0	0	0	0	1
0	1	1	0	1	0	1
1	0	0	1	1	1	0
1	0	1	1	1	1	0
1	1	0	1	1	0	1
1	1	1	1	1	0	1

SEMANTICAL CONSEQUENCE

d) $\{p \vee q,\ q \Rightarrow (r \wedge \overline{p})\} \models q \Leftrightarrow r$ is false.

p	q	r	$p \vee q$	$r \wedge \overline{p}$	$q \Rightarrow (r \wedge \overline{p})$	$q \Leftrightarrow r$
0	0	0	0	0	1	1
0	0	1	0	1	1	0
0	1	0	1	0	0	0
0	1	1	1	1	1	1
1	0	0	1	0	1	1
1	0	1	1	0	1	0
1	1	0	1	0	0	0
1	1	1	1	0	0	1

TAUTOLOGICAL EQUIVALENCE

Definition

We say that a formulas α and β are tautologically equivalent (or semantically equivalent), provided $\alpha \models \beta$ and $\beta \models \alpha$.

The fact that formulas α and β are tautologically equivalent, is denoted by $\alpha \models\!\dashv \beta$.

Since $x \Rightarrow y \models\!\dashv \overline{x} \vee y$ and $x \Leftrightarrow y \models\!\dashv (\overline{x} \vee y) \wedge (x \vee \overline{y})$, we know how to rewrite every formula that contains logical connectives $\Rightarrow$ or $\Leftrightarrow$ into a formula that is equivalent to it and that contains only connectives $\vee$, $\wedge$, $\overline{}$.

TAUTOLOGICAL EQUIVALENCE

Example

Show that the formulas $x \Leftrightarrow y$ and $\overline{(\overline{x} \wedge y)} \wedge (\overline{x} \vee y)$ are tautologically equivalent.

Solution: Let f_1 be a boolean function corresponding to the formula $x \Leftrightarrow y$ and f_2 be a boolean function corresponding to the formula $\overline{(\overline{x} \wedge y)} \wedge (\overline{x} \vee y)$. Let's determine the values of these functions using a truth table.

x	y	f_1	$\overline{x}$	$\overline{x} \wedge y$	$\overline{(\overline{x} \wedge y)}$	$\overline{x} \vee y$	f_2
1	1	1	0	0	1	1	1
1	0	0	0	0	1	0	0
0	1	0	1	1	0	1	0
0	0	1	1	0	1	1	1

From the table we can see that the functions f_1 a f_2 are equal for all $(x, y) \in D^2$. So, a given formulas are tautologically equivalent, so $x \Leftrightarrow y \models\!\!\!= \overline{(\overline{x} \wedge y)} \wedge (\overline{x} \vee y)$.

TAUTOLOGICAL EQUIVALENCE

Theorem

Let α, β, γ are formulas. The following hold:

- $\alpha \wedge \alpha \models\mid \alpha, \qquad \alpha \vee \alpha \models\mid \alpha,$
- *Commutativy of $\wedge$ and $\vee$*
  $\alpha \wedge \beta \models\mid \beta \wedge \alpha, \qquad \alpha \vee \beta \models\mid \beta \vee \alpha,$
- *Associativity of $\wedge$ and $\vee$*
  $\alpha \wedge (\beta \wedge \gamma) \models\mid (\alpha \wedge \beta) \wedge \gamma, \qquad \alpha \vee (\beta \vee \gamma) \models\mid (\alpha \vee \beta) \vee \gamma,$
- *Absorption of $\wedge$ and $\vee$*
  $\alpha \wedge (\beta \vee \alpha) \models\mid \alpha, \qquad \alpha \vee (\beta \wedge \alpha) \models\mid \alpha,$
- *Double negation*
  $\neg\neg\alpha \models\mid \alpha,$
- *De Morgans laws*
  $\neg(\alpha \wedge \beta) \models\mid (\neg\alpha \vee \neg\beta), \qquad \neg(\alpha \vee \beta) \models\mid (\neg\alpha \wedge \neg\beta),$
- *Distributivity laws*
  $\alpha \vee (\beta \wedge \gamma) \models\mid (\alpha \vee \beta) \wedge (\alpha \vee \gamma), \qquad \alpha \wedge (\beta \vee \gamma) \models\mid (\alpha \wedge \beta) \vee (\alpha \wedge \gamma),$

TAUTOLOGICAL EQUIVALENCE

Theorem

If moreover $\boldsymbol{T}$ *is a tautology and* $\boldsymbol{F}$ *is a contradiction, then*

- $\boldsymbol{T} \wedge \alpha \;\models\!\!\!\!\!\models\; \alpha, \qquad \boldsymbol{F} \wedge \alpha \;\models\!\!\!\!\!\models\; \boldsymbol{F},$
- $\boldsymbol{T} \vee \alpha \;\models\!\!\!\!\!\models\; \boldsymbol{T}, \qquad \boldsymbol{F} \vee \alpha \;\models\!\!\!\!\!\models\; \alpha,$
- $\alpha \wedge \neg\alpha \;\models\!\!\!\!\!\models\; \boldsymbol{F}, \qquad \alpha \vee \neg\alpha \;\models\!\!\!\!\!\models\; \boldsymbol{T}.$

Theorem

We have $\alpha \;\models\!\!\!\!\!\models\; \beta$ *if and only if the formula* $\alpha \Leftrightarrow \beta$ *is a tautology.*

DISJUNCTIVE AND CONJUNCTIVE NORMAL FORM

Disjunctive normal form (sum–of–product expansion) of the Boolean function $f(x, y, z)$.
Conjunctive normal form (sum–of–product expansion) of the Boolean function $f(x, y, z)$.

Theorem

For every boolean function f there exists a DNF formula that correspondes to f.
For every formula α there exists a DNF formula β such that $\alpha \models\mathrel{\mkern-6mu}\Dashv \beta$.

Theorem

For every boolean function f there exists a CNF formula that correspondes to f.
For every formula α there exists a CNF formula γ such that $\alpha \models\mathrel{\mkern-6mu}\Dashv \gamma$.

Minimal realization of boolean function

Karnaugh map is based on the reorganized thruthful table of Boolean function.
Karnaugh map in 3-dimensional space is a surface of torus drawn in the plane as a grid of the form 1x2, 2x4, 4x4, 8x4,...
Karnaugh map may be used to quickly eliminate redundant operation in a Boolean function.
Adjacent rows (columns) are different in just one entry.
Bolt rows (columns) are adjacent.

Basic zero matrix (basic unit matrix) is a part of Karnaugh map consisting of zeros (ones) creating a rectangle of adjacent cells. Dimensions of this rectangle are fold of 2.

KARNAUGH MAPS

- Matrix should be as large as possible.
- Every zero (one) in karnaugh map must be in at least one basic zero (unit) matrix.
- The number of basic zero (unit) matrices must be as small as possible.

Minimal disjunctive form of boolean function
Minimal conjunctive form of boolean function