NMPaMŠ – 3.cvičenie

RNDr. Z. Gibová, PhD.

Riešenie sústav nelineárnych rovníc
Newtonova iteračná metóda

sústava rovníc $f_1(x, y) = 0$
$f_2(x, y) = 0$

po úpravách

$$\frac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x}\left(x^{(k+1)} - x^{(k)}\right) + \frac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y}\left(y^{(k+1)} - y^{(k)}\right) = -f_1(x^{(k)}, y^{(k)}),$$

$$\frac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x}\left(x^{(k+1)} - x^{(k)}\right) + \frac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y}\left(y^{(k+1)} - y^{(k)}\right) = -f_2(x^{(k)}, y^{(k)}).$$

Neznáme $\left(x^{(k+1)} - x^{(k)}\right)$, $\left(y^{(k+1)} - y^{(k)}\right)$, k – iterácia (k = 0, 1, 2, ...)

Cramerovo pravidlo pre riešenie sústavy

$$W_1^{(k)} = \begin{vmatrix} -f_1(x^{(k)}, y^{(k)}) & \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y} \\ -f_2(x^{(k)}, y^{(k)}) & \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y} \end{vmatrix}, \quad W_2^{(k)} = \begin{vmatrix} \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x} & -f_1(x^{(k)}, y^{(k)}) \\ \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x} & -f_2(x^{(k)}, y^{(k)}) \end{vmatrix},$$

Cramerovo pravidlo pre riešenie sústavy

$$\frac{\partial f_1(x^{(k)},y^{(k)})}{\partial x}(x^{(k+1)}-x^{(k)})+\frac{\partial f_1(x^{(k)},y^{(k)})}{\partial y}(y^{(k+1)}-y^{(k)})=-f_1(x^{(k)},y^{(k)}),$$

$$\frac{\partial f_2(x^{(k)},y^{(k)})}{\partial x}(x^{(k+1)}-x^{(k)})+\frac{\partial f_2(x^{(k)},y^{(k)})}{\partial y}(y^{(k+1)}-y^{(k)})=-f_2(x^{(k)},y^{(k)}).$$

$$W^{(k)}=\begin{vmatrix} \dfrac{\partial f_1(x^{(k)},y^{(k)})}{\partial x} & \dfrac{\partial f_1(x^{(k)},y^{(k)})}{\partial y} \\ \dfrac{\partial f_2(x^{(k)},y^{(k)})}{\partial x} & \dfrac{\partial f_2(x^{(k)},y^{(k)})}{\partial y} \end{vmatrix}$$

Pr. 1 Newtonovou iteračnou metódou s presnosťou $\varepsilon = 10^{-2}$ riešte v danej oblasti $D = \langle 1,2 \rangle \times \langle 0,1 \rangle$ sústavu nelineárnych rovníc $xy - x + 1 = 0$, $2x^2 + y - 3 = 0$.

1. **Zapíšeme sústavu rovníc $f_1(x, y)$, $f_2(x, y)$, určíme $-f_1(x, y)$, $-f_2(x, y)$, urobíme ich parciálne derivácie podľa *x* a *y***

$$f_1(x,y) = xy - x + 1 \qquad -f_1(x,y) = -xy + x - 1$$

$$f_2(x,y) = 2x^2 + y - 3 \qquad -f_2(x,y) = -2x^2 - y + 3$$

$$\frac{\partial f_1(x,y)}{\partial x} = y - 1 \qquad \frac{\partial f_1(x,y)}{\partial y} = x$$

$$\frac{\partial f_2(x,y)}{\partial x} = 4x \qquad \frac{\partial f_2(x,y)}{\partial y} = 1$$

2. **Určíme počiatočné hodnoty $x^{(0)}$, $y^{(0)}$ z oblasti D**

$$\boldsymbol{x^{(0)} = 1, y^{(0)} = 0}$$

3. Pomocou Cramerovho pravidla určíme determinanty $W^{(k)}$, $W_1^{(k)}$, $W_2^{(k)}$

$$W_1^{(k)} = \begin{vmatrix} -f_1(x^{(k)}, y^{(k)}) & \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y} \\ -f_2(x^{(k)}, y^{(k)}) & \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y} \end{vmatrix}$$

$$W_1{}^{(k)} = \begin{vmatrix} -xy + x - 1 & x \\ -2x^2 - y + 3 & 1 \end{vmatrix}$$

$$W_2^{(k)} = \begin{vmatrix} \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x} & -f_1(x^{(k)}, y^{(k)}) \\ \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x} & -f_2(x^{(k)}, y^{(k)}) \end{vmatrix}$$

$$W_2{}^{(k)} = \begin{vmatrix} y - 1 & -xy + x - 1 \\ 4x & -2x^2 - y + 3 \end{vmatrix}$$

$$W^{(k)} = \begin{vmatrix} \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x} & \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y} \\ \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x} & \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y} \end{vmatrix}$$

$$W^{(k)} = \begin{vmatrix} y - 1 & x \\ 4x & 1 \end{vmatrix}$$

3. Pomocou Cramerovho pravidla určíme determinanty $W^{(k)}$, $W_1^{(k)}$, $W_2^{(k)}$

pre k = 0

$$W_1{}^{(0)} = \begin{vmatrix} 0+1-1 & 1 \\ -2+3 & 1 \end{vmatrix} = -1$$

$$W_2{}^{(0)} = \begin{vmatrix} 0-1 & 0 \\ 4.1 & 1 \end{vmatrix} = -1$$

$$W^{(0)} = \begin{vmatrix} 0-1 & 1 \\ 4.1 & 1 \end{vmatrix} = -1-4 = -5$$

4. Hodnoty neznámych $x^{(k+1)}$, $y^{(k+1)}$ sú určené interačným procesom

$$x^{(k+1)} = x^{(k)} + \frac{W_1^{(k)}}{W^{(k)}}$$

$$x^{(1)} = x^{(0)} + \frac{W_1{}^{(0)}}{W^{(0)}} = 1 + \frac{-1}{-5} = 1{,}2$$

$$y^{(k+1)} = y^{(k)} + \frac{W_2^{(k)}}{W^{(k)}}$$

$$y^{(1)} = y^{(0)} + \frac{W_2{}^{(0)}}{W^{(0)}} = 0 + \frac{-1}{-5} = 0{,}2$$

5. Hodnoty zapíšeme do tabuľky

k	$x^{(k)}$	$y^{(k)}$	$W_1{}^{(k)}$	$W_2{}^{(k)}$	$W^{(k)}$	$\|\overline{x}^{(k+1)} - \overline{(k)}\| < \varepsilon$
0	1	0	-1	-1	-5	**0,2** $> \varepsilon$
1	1,2	0,2				

6. Iteračný proces ukončíme

a) ak je daná presnosť ε

$$\|\overline{x}^{(k+1)} - \overline{(k)}\| = \max\{|x^{(k+1)} - x^{(k)}|, |y^{(k+1)} - y^{(k)}|\} < \varepsilon$$

$$max\{|1{,}2 - 1|, |0{,}2 - 0|\} = max\{|0{,}2|, |0{,}2|\} = 0{,}2 > \mathbf{0{,}01} \quad \boldsymbol{pokračujeme\ v\ iterácii}$$

b) ak je daný počet iterácií *k*

potom chyba výpočtu $\max\{|x^{(k+1)} - x^{(k)}|, |y^{(k+1)} - y^{(k)}|\}$

Pomocou Cramerovho pravidla určíme determinanty $W^{(k)}$, $W_1^{(k)}$, $W_2^{(k)}$

pre k = 1

$$W_1^{(1)} = \begin{vmatrix} -1{,}2.0{,}2 + 1{,}2 - 1 & 1{,}2 \\ -2.\,1{,}2^2 - 0{,}2 + 3 & 1 \end{vmatrix} = \begin{vmatrix} -0{,}04 & 1{,}2 \\ -0{,}08 & 1 \end{vmatrix} = 0{,}056$$

$$W_2^{(1)} = \begin{vmatrix} 0{,}2 - 1 & -1{,}2.0{,}2 + 1{,}2 - 1 \\ 4.1{,}2 & -2.\,1{,}2^2 - 0{,}2 + 3 \end{vmatrix} = 0{,}256$$

$$W^{(1)} = \begin{vmatrix} 0{,}2 - 1 & 1{,}2 \\ 4.1{,}2 & 1 \end{vmatrix} = -6{,}56$$

Hodnoty neznámych $x^{(k+1)}$, $y^{(k+1)}$ sú určené interačným procesom

$$x^{(2)} = x^{(1)} + \frac{W_1^{(1)}}{W^{(1)}} = 1{,}2 + \frac{0{,}056}{-6{,}56} = 1{,}1915$$

$$y^{(2)} = y^{(1)} + \frac{W_2^{(1)}}{W^{(1)}} =,20 + \frac{0{,}025}{-6{,}56} = 0{,}1610$$

Hodnoty zapíšeme do tabuľky

k	$x^{(k)}$	$y^{(k)}$	$W_1^{(k)}$	$W_2^{(k)}$	$W^{(k)}$	$\|\overline{x}^{(k+1)} - \overline{(k)}\| < \varepsilon$
0	1	0	-1	-1	-5	$0{,}2 > \varepsilon$
1	1,2	0,2	0,056	0,256	-6,56	**0,039** $> \varepsilon$
2	1,1915	0,1610				

$$max\{|1{,}1915 - 1{,}2|, |0{,}161 - 0{,}2|\} = max\{|0{,}0085|, |0{,}039|\} = 0{,}039 > \mathbf{0,01}$$

pokračujeme v iterácii

Pomocou Cramerovho pravidla určíme determinanty $W^{(k)}$, $W_1^{(k)}$, $W_2^{(k)}$

pre k = 2

$$W_1^{(2)} = \begin{vmatrix} -0{,}00033 & 1{,}1915 \\ -0{,}00034 & 1 \end{vmatrix} = 0{,}00007511$$

$$W_2^{(2)} = \begin{vmatrix} -0{,}839 & -0{,}00033 \\ 4{,}766 & -0{,}00034 \end{vmatrix} = 0{,}00185804$$

$$W^{(2)} = \begin{vmatrix} -0{,}839 & 1{,}1915 \\ 4{,}766 & 1 \end{vmatrix} = -6{,}5617689$$

Hodnoty neznámych $x^{(k+1)}$, $y^{(k+1)}$ sú určené interačným procesom

$$x^{(3)} = x^{(2)} + \frac{W_1^{(2)}}{W^{(1)}} = 1{,}1915 + \frac{0{,}00007511}{-6{,}5617689} = 1{,}19149$$

$$y^{(3)} = y^{(2)} + \frac{W_2^{(2)}}{W^{(2)}} = 0{,}161 + \frac{0{,}00185804}{-6{,}5617689} = 0{,}16071$$

Hodnoty zapíšeme do tabuľky

k	$x^{(k)}$	$y^{(k)}$	$W_1^{(k)}$	$W_2^{(k)}$	$W^{(k)}$	$\|\overline{x}^{(k+1)} - \overline{(k)}\| < \varepsilon$
0	1	0	-1	-1	-5	$0{,}2 > \varepsilon$
1	1,2	0,2	0,056	0,256	-6,56	**0,039** $> \varepsilon$
2	1,1915	0,1610	0,00007511	0,00185804	−6,5617689	**0,0029** $< \varepsilon$
3	1,19149	0,16071				

$$max\{|1{,}19149 - 1{,}1915|, |0{,}16071 - 0{,}161|\} = max\{|0{,}00001|, |0{,}00029|\} = 0{,}0029 < \mathbf{0{,}01}$$

koniec iterácie

7. Aproximujeme neznáme hodnotami $x^{(K)}$, $y^{(K)}$

$x^{(3)} = 1{,}191499,$ $\quad x \approx 1{,}19$

$y^{(3)} = 0{,}16071,$ $\quad y \approx 0{,}16$ podľa presnosti $\varepsilon = 0{,}01$

Pr. 2

Newtonovou iteračnou metódou riešte danú sústavu rovníc: sin y – x -1,2 = 0

2y + cos x – 1,7 = 0

z počiatočného bodu (-1;0,2) hodnoty sú dané v radiánoch. Urobte jednu iteráciu a určte chybu výpočtu.

$$f_1(x,y) = \sin y - x - 1{,}2 \qquad -f_1(x,y) = -\sin y + x + 1{,}2$$

$$f_2(x,y) = 2y + \cos x \ - 1{,}7 \qquad -f_2(x,y) = -2y - \cos x + 1{,}7$$

$$\frac{\partial f_1(x,y)}{\partial x} = -1 \qquad \frac{\partial f_1(x,y)}{\partial y} = \cos y$$

$$\frac{\partial f_2(x,y)}{\partial x} = -\sin x \qquad \frac{\partial f_2(x,y)}{\partial y} = 2$$

$$W^{(k)} = \begin{vmatrix} -1 & \cos y \\ -\sin x & 2 \end{vmatrix}$$

$$\boldsymbol{x^{(0)} = -1, y^{(0)} = 0,2}$$

$$W^{(0)} = \begin{vmatrix} -1 & \cos 0{,}2 \\ -\sin(-1) & 2 \end{vmatrix} = -2 - 0{,}8247 = -2{,}8247$$

$$W_1{}^{(k)} = \begin{vmatrix} -1 & -\sin y + x + 1{,}2 \\ -\sin x & -2y - \cos x + 1{,}7 \end{vmatrix}$$

$$W_1{}^{(0)} = \begin{vmatrix} -1 & 0{,}00133 \\ 0{,}8415 & 0{,}75969 \end{vmatrix} = -0{,}7609$$

$$W_2{}^{(k)} = \begin{vmatrix} -\sin y + x + 1{,}2 & \cos y \\ -2y - \cos x + 1{,}7 & 2 \end{vmatrix}$$

$$W_2{}^{(k)} = \begin{vmatrix} 0{,}00133 & 0{,}98006 \\ 0{,}75969 & 2 \end{vmatrix} = -0{,}7419$$

$$x^{(1)} = x^{(0)} + \frac{W_1{}^{(0)}}{W^{(0)}} = -1 + \frac{-0{,}7419}{-2{,}8247} = -0{,}7374$$

$$y^{(1)} = y^{(0)} + \frac{W_2{}^{(0)}}{W^{(0)}} = 0{,}2 + \frac{-0{,}7609}{-2{,}8247} = 0{,}4693$$

k	$x^{(k)}$	$y^{(k)}$	$W_1{}^{(k)}$	$W_2{}^{(k)}$	$W^{(k)}$
0	- 1	0,2	- 0,7419	- 0,7608	- 2,8247
1	- 0,7374	0,4693	0,0018	- 0,0281	- 2,599

$$\max\{|x^{(k+1)} - x^{(k)}|, |y^{(k+1)} - y^{(k)}|\}$$

$$max\{|-0{,}7374 + 1|, |0{,}4693 - 0{,}2|\} = max\{|-0{,}2626|, |0{,}2693|\} = 0{,}2693$$

$x \approx -0{,}7374, y \approx 0{,}4693$ s chybou 0,2693

Dú:

3.8.

Newtonovou metódou s presnosťou $\varepsilon = 10^{-4}$ riešte v oblasti $D = \langle 1;1,2 \rangle \times \langle 0,5;1 \rangle$ sústavu nelineárnych rovníc $9x^2 + y^2 - 10 = 0$, $2xy + 4y - 5 = 0$.

[1,0173; 0,8286]

3.7.

Newtonovou metódou s presnosťou $\varepsilon = 10^{-3}$ riešte v oblasti $D = \langle 1,5;2 \rangle \times \langle 0,5;1 \rangle$ sústavu nelineárnych rovníc $xy - 1 = 0$, $y - (x-1)^3 - 0,21 = 0$.

[1,719; 0,582]

Preskúšajte sa

Vyberte správne tvrdenia.

1. Pri Newtonovej iteračnej metóde pri riešení sústavy dvoch nelineárnych rovníc je determinant $W_1^{(k)}$ určený

a) $\begin{vmatrix} -f_1(x^{(k)}, y^{(k)}) & \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y} \\ -f_2(x^{(k)}, y^{(k)}) & \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y} \end{vmatrix}$

b) $\begin{vmatrix} \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x} & -f_1(x^{(k)}, y^{(k)}) \\ \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x} & -f_2(x^{(k)}, y^{(k)}) \end{vmatrix}$

c) $\begin{vmatrix} \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x} & \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y} \\ \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x} & \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y} \end{vmatrix}$

2. Pri Newtonovej iteračnej metóde pri riešení sústavy dvoch nelineárnych rovníc je determinant $W_2^{(k)}$ určený

a) $\begin{vmatrix} \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x} & -f_1(x^{(k)}, y^{(k)}) \\ \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x} & -f_2(x^{(k)}, y^{(k)}) \end{vmatrix}$

b) $\begin{vmatrix} \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x} & \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y} \\ \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x} & \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y} \end{vmatrix}$

c) $\begin{vmatrix} -f_1(x^{(k)}, y^{(k)}) & \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y} \\ -f_2(x^{(k)}, y^{(k)}) & \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y} \end{vmatrix}$

3. Pri Newtonovej iteračnej metóde pri riešení sústavy dvoch nelineárnych rovníc je determinant $W^{(k)}$ určený

a) $\begin{vmatrix} -f_1(x^{(k)}, y^{(k)}) & \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y} \\ -f_2(x^{(k)}, y^{(k)}) & \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y} \end{vmatrix}$

b) $\begin{vmatrix} \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x} & -f_1(x^{(k)}, y^{(k)}) \\ \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x} & -f_2(x^{(k)}, y^{(k)}) \end{vmatrix}$

c) $\begin{vmatrix} \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial x} & \dfrac{\partial f_1(x^{(k)}, y^{(k)})}{\partial y} \\ \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial x} & \dfrac{\partial f_2(x^{(k)}, y^{(k)})}{\partial y} \end{vmatrix}$

4. Iteračný proces pri Newtonovej iteračnej metóde je daný

a) $$x^{(k+1)} = x^{(k)} - \frac{{W_1}^{(k)}}{W^{(k)}} \qquad y^{(k+1)} = y^{(k)} - \frac{{W_2}^{(k)}}{W^{(k)}}$$

b) $$x^{(k+1)} = x^{(k)} + \frac{{W_2}^{(k)}}{W^{(k)}} \qquad y^{(k+1)} = y^{(k)} + \frac{{W_1}^{(k)}}{W^{(k)}}$$

c) $$x^{(k+1)} = x^{(k)} + \frac{{W_1}^{(k)}}{W^{(k)}} \qquad y^{(k+1)} = y^{(k)} + \frac{{W_2}^{(k)}}{W^{(k)}}$$

5. Ak podmienka $\|\overline{x}^{(k+1)} - \overline{(k)}\| = \max\{|x^{(k+1)} - x^{(k)}|, |y^{(k+1)} - y^{(k)}|\} < \varepsilon$ potom

a) pokračujeme v iteračnom procese,
b) iteračný proces ukončíme,
c) vypočítame chybu výpočtu.

6. Ak urobíme pri výpočte sústavy dvoch rovníc dve iterácie, potom neznáme veličiny aproximujeme

a) hodnotami $x^{(1)}$, $y^{(1)}$,
b) hodnotami $x^{(2)}$, $y^{(2)}$,
c) hodnotami $x^{(3)}$, $y^{(3)}$.

Správne odpovede:
1a, 2a, 3c, 4c, 5b, 6b

Hodnotenie:
0 – 1 nesprávne odpovede – máš vedomosti o metóde riešenia sústavy nelineárnych rovníc
2 nesprávne odpovede – tvoje vedomosti sú celkom dobré
3 a viac nesprávnych odpovedí – tvoje vedomosti nie sú postačujúce, odporúčam sa na to ešte raz pozrieť