Matematika 1 – 4.cvičenie

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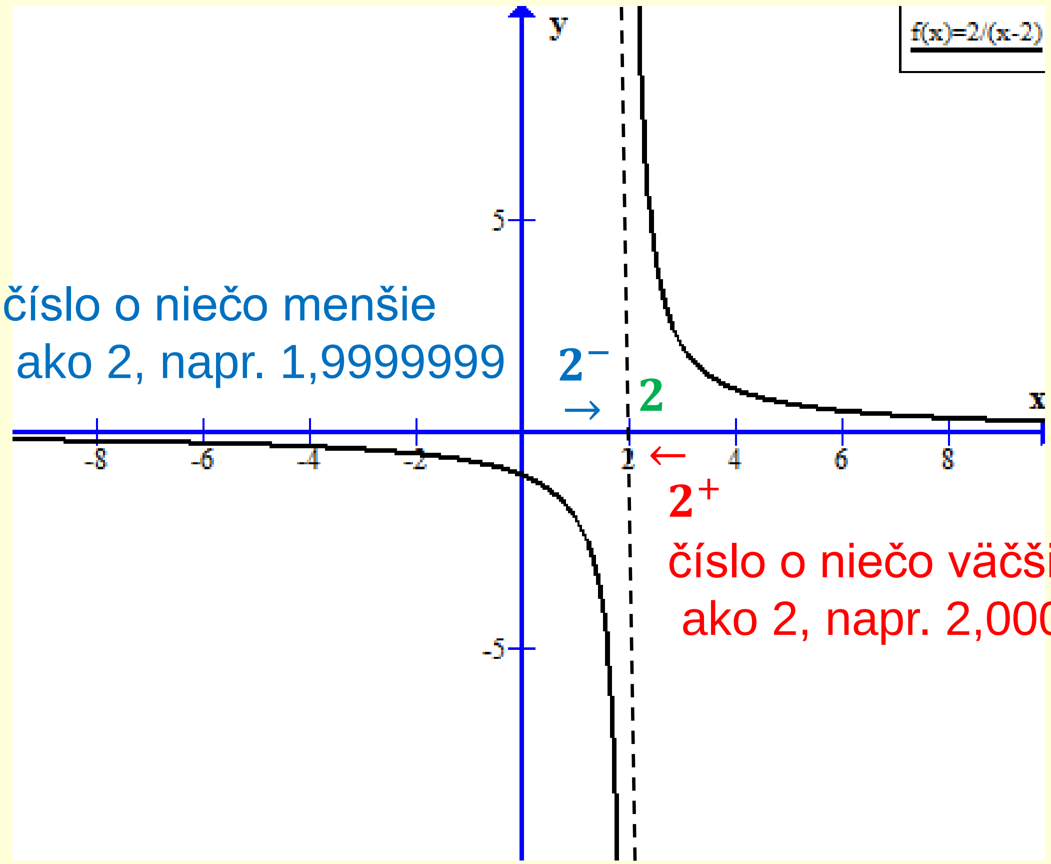
Asymptoty funkcie

Jednostranné limity:

limita pre *x* idúce k ***a*** sprava $\lim_{x \to a^+} f(x)$

limita pre *x* idúce k ***a*** zľava $\lim_{x \to a^-} f(x)$



číslo o niečo menšie
ako 2, napr. 1,9999999

2^- →

2

← 2^+

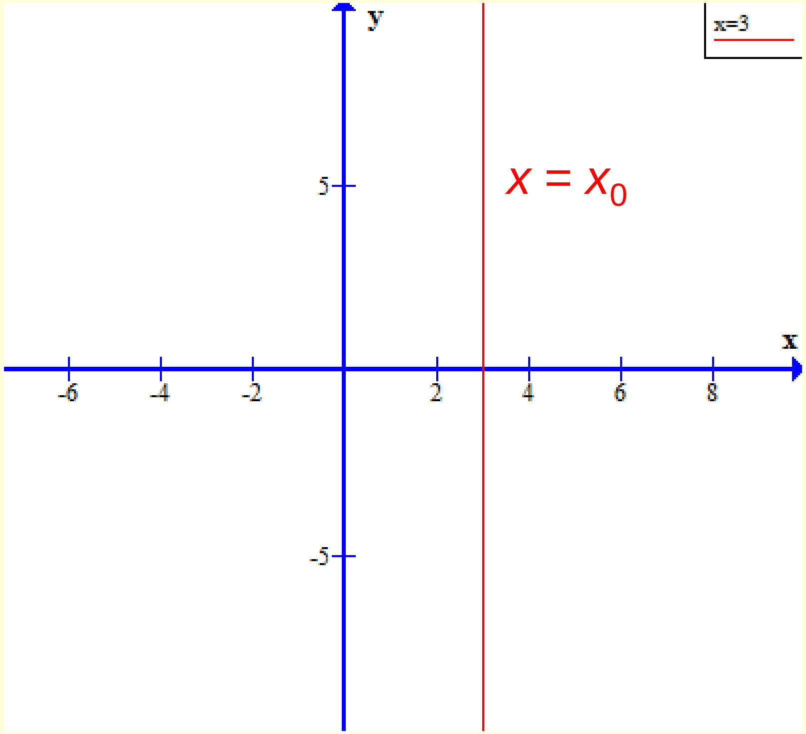
číslo o niečo väčšie
ako 2, napr. 2,0000000001

Asymptota je priamka, ktorá opisuje správanie sa krivky.

Asymptota bez smernice (ABS): vypočítame jednostranné limity v bode nespojitosti x_0 (bod v ktorom funkcie nie je definovaná), ak vyjdú nevlastné čísla $\pm\infty$ (stačí jedna z limít rovná $\pm\infty$, potom funkcia má ABS)

$$\lim_{x \to x_0^+} f(x) = \pm\infty \qquad \lim_{x \to x_0^-} f(x) = \pm\infty$$

potom ABS je $x = x_0$



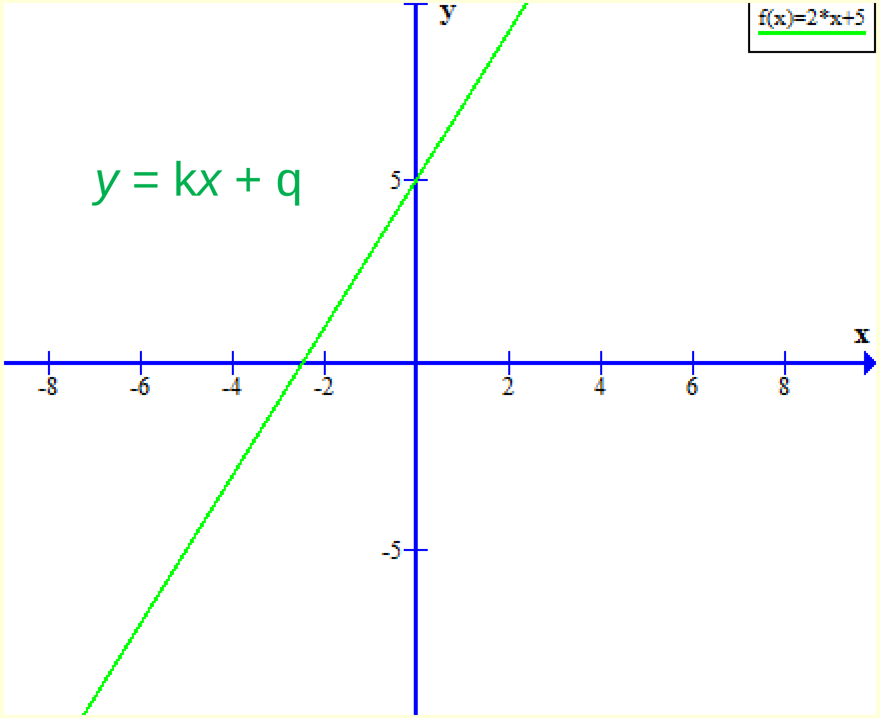
Asymptota so smernicou (ASS): priamka $y = kx + q$

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x}$$

$$q = \lim_{x \to \pm\infty} [f(x) - kx]$$

vypočítame k, q, ak vyjdú vlastné čísla, potom

ASS je $y = k_1 x + q_1$, $y = k_2 x + q_2$



Pr.1: Určte ASS a ABS funkcie

$$y = \frac{3(x-1)^2}{x-2}$$

1. Určiť D(f) a bod nespojitosti.

$$x - 2 \neq 0$$

$$x \neq 2\,,\quad D(f) = R - \{2\}$$

bod nespojitosti $\boldsymbol{x_0 = 2}$

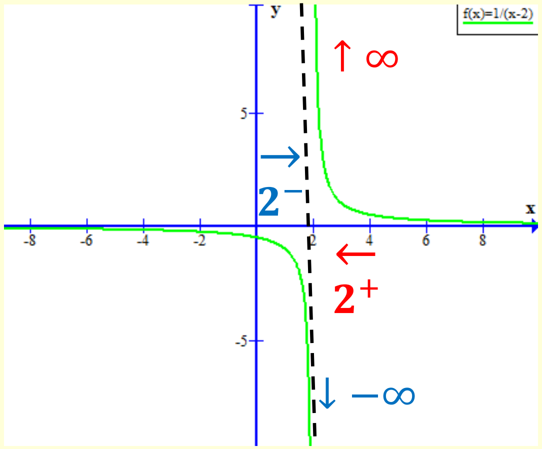
2. Určiť ABS v bode nespojitosti

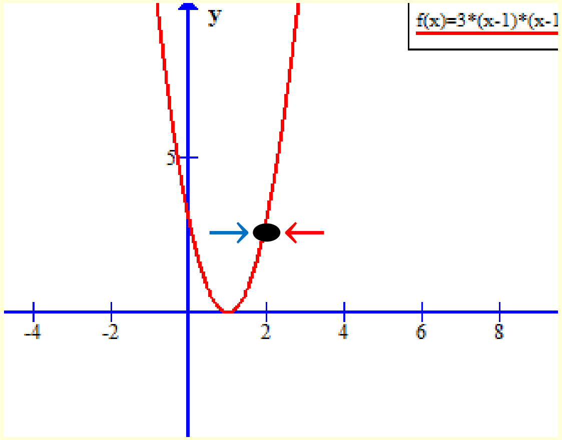
$$\frac{1}{2^+ - 2} = \frac{1}{0^+} \qquad 3(2^+ - 1)^2 = +\,číslo$$

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to 2^+} \frac{3(x-1)^2}{x-2} = \lim_{x\to 2^+} \frac{1}{x-2} \cdot \lim_{x\to 2^+} 3(x-1)^2 = +\infty$$

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to 2^-} \frac{3(x-1)^2}{x-2} = \lim_{x\to 2^-} \frac{1}{x-2} \cdot \lim_{x\to 2^-} 3(x-1)^2 = -\infty$$

$$\frac{1}{2^- - 2} = \frac{1}{0^-} \qquad 3(2^- - 1)^2 = +\,číslo$$





ABS: $x = 2$

3. Určiť ASS

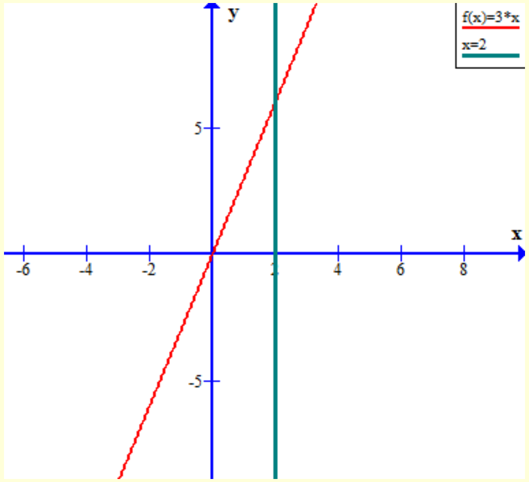
$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{3(x-1)^2}{x(x-2)} = \lim_{x\to\infty} \frac{3x^2-6x+3}{x^2-2x} = 3$$

$$k = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{3(x-1)^2}{x(x-2)} = \lim_{x\to-\infty} \frac{3x^2-6x+3}{x^2-2x} = 3$$

$$q = \lim_{x\to\infty} [f(x)-kx] = \lim_{x\to\infty} \left[\frac{3(x-1)^2}{(x-2)} - 3x\right] = \lim_{x\to\infty} \left[\frac{3x^2-6x+3-3x^2+6x}{(x-2)}\right]$$

$$= \lim_{x\to\infty} \left[\frac{3}{(x-2)}\right] = 0 \qquad q = \lim_{x\to-\infty} \left[\frac{3}{(x-2)}\right] = 0$$

ASS: $y = 3x \;\; pre \;\; x \to \pm\infty$



ASS: $y = 3x \quad pre \;\; x \to \pm\infty$

ABS: $x = 2$

Pr.2: 47/ 2 Určte ASS a ABS funkcie 1. Určiť D(f) a bod nespojitosti.

$$y = x^3 + 3x^2 - 2$$

$$D(f) = R$$

bod nespojitosti $\boldsymbol{x_0}$ nemá

2. Určiť ABS v bode nespojitosti – ABS nemá, lebo nemá bod nespojitosti

3. Určiť ASS

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^3 + 3x^2 - 2}{x} = \lim_{x \to \pm\infty} (x^2 + 3x - \frac{2}{x}) = \pm\infty$$

$$q = \lim_{x \to \infty} [f(x) - kx] \; - nerátame, k \; sa \; nerovná \; vlastnému \; číslu$$

ASS nemá

Pr.3: Určte ASS a ABS funkcie

$$y = \frac{2x^2}{2x-1}$$

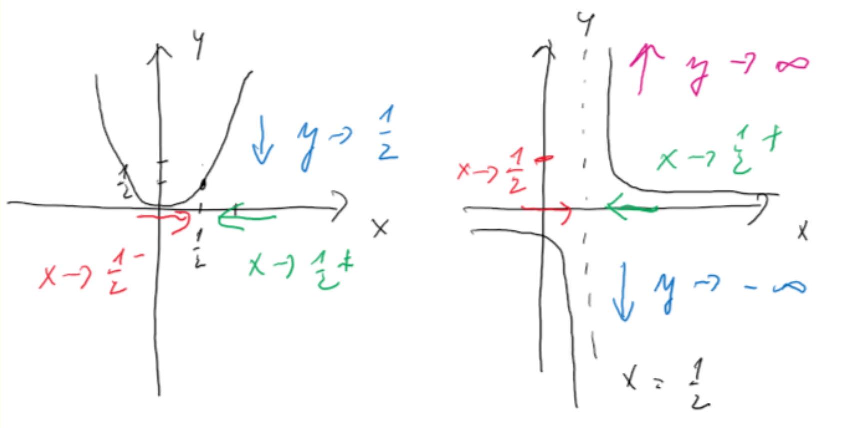
$2x - 1 \neq 0$

$x \neq \frac{1}{2}$, $D(f) = R - \left\{\frac{1}{2}\right\}$

bod nespojitosti $\boldsymbol{x_0 = 0,5}$

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to 0{,}5^+} \frac{2x^2}{2x-1} = \lim_{x\to 0{,}5^+} \frac{1}{2x-1} \cdot \lim_{x\to 0{,}5^+} 2x^2 = \frac{1}{0^+} \cdot (+\, \check{c}\acute{\imath}slo) = +\infty$$

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to 0{,}5^-} \frac{2x^2}{2x-1} = \lim_{x\to 0{,}5^-} \frac{1}{2x-1} \cdot \lim_{x\to 0{,}5^-} 2x^2 = \frac{1}{0^-} \cdot (+\, \check{c}\acute{\imath}slo) = -\infty$$

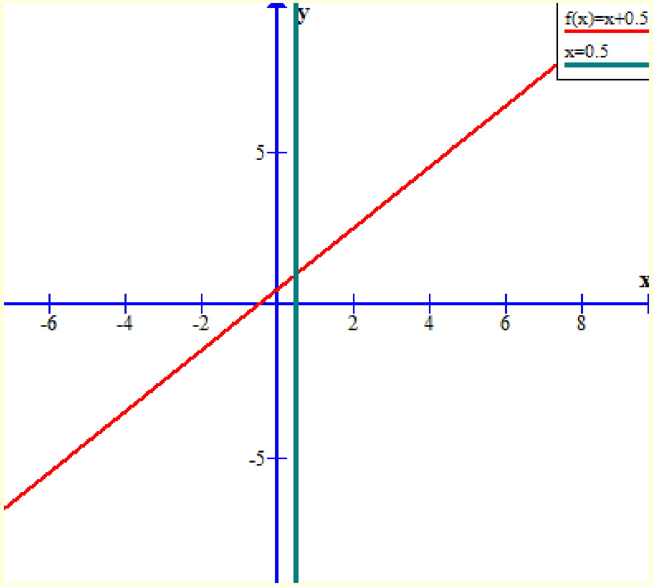


ABS: $x = 0{,}5$

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\frac{2x^2}{2x-1}}{x} = \lim_{x\to\pm\infty} \frac{2x^2}{2x^2-x} = 1$$

$$q = \lim_{x\to\pm\infty} [f(x) - kx] = \lim_{x\to\pm\infty} \left[\frac{2x^2}{2x-1} - x\right] = \lim_{x\to\pm\infty} \left[\frac{2x^2-2x^2+x}{2x-1}\right] = \frac{1}{2}$$

ASS: $y = x + 0{,}5 \quad pre\ \ x \to \pm\infty$



ASS: $y = x + 0{,}5 \quad pre\ \ x \to \pm\infty$

ABS: $x = 0{,}5$

Pr.4: 47 / 12 Určte ASS a ABS funkcie

$$y = \frac{x^2}{4 - x^2}$$

$$4 - x^2 \neq 0$$
$$x^2 \neq 4$$
$$|x| \neq 2, x \neq \pm 2, D(f) = R - \{\pm 2\}$$

bod nespojitosti $\boldsymbol{x_0 = -2, 2}$

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to 2^+} \frac{x^2}{4 - x^2} = \lim_{x \to 2^+} \frac{1}{4 - x^2} . \lim_{x \to 2^+} x^2 = \frac{1}{0^-} . 4^+ = -\infty$$

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to 2^-} \frac{x^2}{4 - x^2} = \lim_{x \to 2^-} \frac{1}{4 - x^2} . \lim_{x \to 2^-} x^2 = \frac{1}{0^+} . 4^+ = \infty$$

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to -2^+} \frac{x^2}{4 - x^2} = \lim_{x \to -2^+} \frac{1}{4 - x^2} . \lim_{x \to -2^+} x^2 = \frac{1}{0^+} . 4^- = \infty$$

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to -2^-} \frac{x^2}{4 - x^2} = \lim_{x \to -2^-} \frac{1}{4 - x^2} . \lim_{x \to -2^-} x^2 = \frac{1}{0^-} . 4^- = -\infty$$

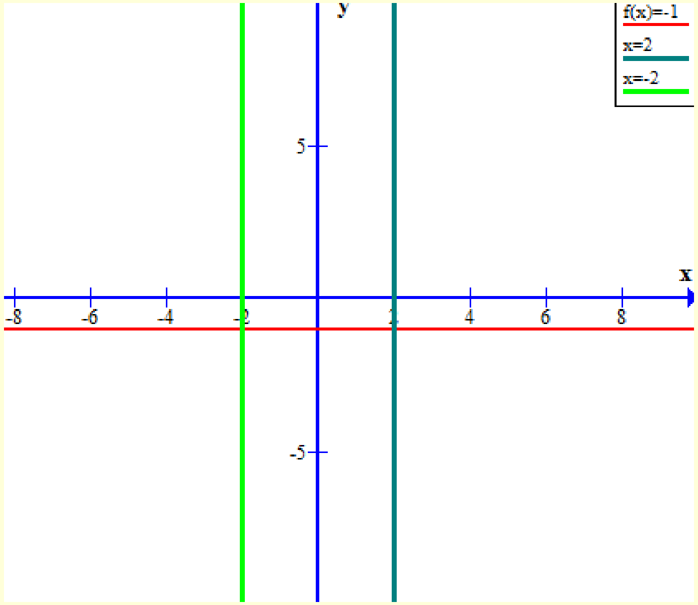
ABS: $x = 2$

ABS: $x = -2$

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2}{x(4-x^2)} = \lim_{x\to\infty} \frac{x^2}{4x-x^3} = 0$$

$$q = \lim_{x\to\infty} [f(x) - kx] = \lim_{x\to\infty} \left[\frac{x^2}{4-x^2}\right] = -1$$

ASS: $y = -1 \; pre \;\; x \to \pm\infty$



ASS: $y = -1 \;\; pre \;\; x \to \pm\infty$

ABS: $x = -2$

ABS: $x = 2$

Pr.5: 47 / 11 Určte ASS a ABS funkcie

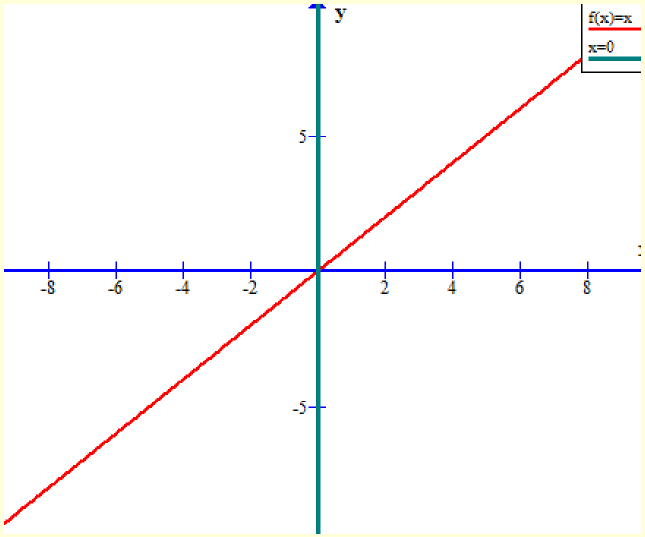
$$x \neq 0$$
$$D(f) = R - \{0\}$$

$$y = x - \frac{1}{x}$$

bod nespojitosti $x_0 = 0$

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to 0^+} x - \frac{1}{x} = 0^+ - \frac{1}{0^+} = -\infty$$

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to 0^-} x - \frac{1}{x} = 0^- - \frac{1}{0^-} = \infty$$



$$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x - \frac{1}{x}}{x} = \lim_{x \to \infty} \frac{x^2 - 1}{x^2} = 1$$

ABS: $x = 0$

ASS: $y = x \;\; pre \;\; x \to \pm\infty$

$$q = \lim_{x \to \infty} [f(x) - kx] = \lim_{x \to \infty} \left[x - \frac{1}{x} - 1.x\right] = \lim_{x \to \infty} \left[-\frac{1}{x}\right] = 0$$

Dú – 47 / 1,5,6,8,9,12,13,16

Testík: Vyberte správne tvrdenia.

1. Asymptotu bez smernice určujeme
   a) v ľubovoľnom bode z definičného odboru funkcie,
   b) v bode nespojitosti funkcie,
   c) v žiadnom bode.

2. Funkcia má asymptotu so smernicou, ak limity $\lim_{x\to\pm\infty}\frac{f(x)}{x}$ a $\lim_{x\to\pm\infty}[f(x)-kx]$
   a) sú vlastné čísla,
   b) sú nevlastné čísla,
   c) sú rovné nule.

3. Funkcia $y=\frac{2}{x}$
   a) má asymptotu bez smernice v bode 0,
   b) má asymptotu bez smernice v bode 2,
   c) nemá asymptotu bez smernice.

4. Ak k = 2 a q = -1, potom zápis asymptoty so smernicou je
   a) y = 2x + 1,
   b) y = -1x + 2,
   c) y = 2x – 1,
   d) y = 2.

Derivácia funkcie

Označenie derivácie funkcie v bode x_0: f ´(x_0)

Pravidlá pre výpočet derivácie funkcie**:**

$$[cf(x)]' = cf'(x) \qquad c \in \mathrm{R}$$

$$[f(x)+g(x)]' = f'(x)+g'(x)$$

$$[f(x)g(x)]' = f'(x)g(x)+f(x)g'(x)$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}.$$

$$[f(g(x))]' = f'(g(x))g'(x).$$

$$\left[f(x)^{g(x)}\right]' = \left[e^{g(x)\cdot\ln f(x)}\right]'$$

Napr.:

$(3x)'$

$(2x^2+5-\ln x)'$

$(x\cdot\ln x)'$

$\left(\frac{3x}{\sin x}\right)'$

$(\ln \sin 2x)'$

$\left[(\sin x)^x = e^{x\ln\sin x}\right]'$

Derivácie elementárnych funkcií:

- $[c]' = 0$
- $[x^\alpha]' = \alpha x^{\alpha-1}, \alpha \in R$
- $[\sin x]' = \cos x$
- $[\cos x]' = -\sin x$
- $[\operatorname{tg} x]' = \frac{1}{\cos^2 x}$
- $[\operatorname{cotg} x]' = -\frac{1}{\sin^2 x}$
- $[\arcsin x]' = \frac{1}{\sqrt{1-x^2}}$

$(x^1)' = 1$

- $[\arccos x]' = -\frac{1}{\sqrt{1-x^2}}$
- $[\operatorname{arctg} x]' = \frac{1}{1+x^2}$
- $[\operatorname{arccotg} x]' = -\frac{1}{1+x^2}$
- $[e^x]' = e^x$
- $[a^x]' = a^x \ln a$
- $[\ln x]' = \frac{1}{x}$
- $[\log_a x]' = \frac{1}{x \ln a}$

Pr.1: Vypočítajte deriváciu funkcie $f(x) = \frac{\sin x}{2} + x.2^x + \frac{x}{\ln x} - \sqrt{x^3}$

$$f\,'(x) = \frac{(\sin x)'}{2} + (x.2^x)' + \left(\frac{x}{\ln x}\right)' - \left(x^{\frac{3}{2}}\right)'$$

$$f\,'(x) = \frac{1}{2}(\sin x)' + x'.2^x + x.(2^x)' + \frac{x'\ln x - x(\ln x)'}{(\ln x)^2} - \left(x^{\frac{3}{2}}\right)'$$

$$(ab)' = a'b + ab'$$

$$\left(\frac{a}{b}\right)' = \frac{a'b - ab'}{b^2} \qquad \left(x^{\frac{3}{2}}\right)' = \frac{3}{2}x^{\frac{1}{2}}$$

$$(x^1)' = 1.x^0 = 1 \qquad (2^x)' = 2^x.\ln 2$$

$$f\,'(x) = \frac{1}{2}\cos x + 1.2^x + x.2^x \ln 2 + \frac{1\ln x - x\frac{1}{x}}{(\ln x)^2} - \frac{3}{2}x^{\frac{3}{2}-1}$$

$$f\,'(x) = \frac{1}{2}\cos x + 2^x + x.2^x \ln 2 + \frac{1\ln x - 1}{(\ln x)^2} - \frac{3}{2}x^{\frac{1}{2}}$$

$$f\,'(x) = \frac{1}{2}\cos x + 2^x + x.2^x \ln 2 + \frac{1\ln x - 1}{(\ln x)^2} - \frac{3}{2}\sqrt{x}$$

Pr.2: 25 /2 Vypočítajte deriváciu funkcie $f(x)=\sqrt[3]{x^4}+5^x-\ln x$

$$f(x)=\sqrt[3]{x^4}+5^x-\ln x = x^{\frac{4}{3}}+5^x-\ln x$$

$$f'(x)=\left(x^{\frac{4}{3}}\right)'+\left(5^x\right)'-\left(\ln x\right)'$$

$$f'(x)=\frac{4}{3}x^{\frac{4}{3}-1}+5^x\ln 5-\frac{1}{x}$$

$$(\ln x)'=\frac{1}{x}$$

$$f'(x)=\frac{4}{3}x^{\frac{1}{3}}+5^x\ln 5-\frac{1}{x}=\frac{4}{3}\sqrt[3]{x}+5^x\ln 5-\frac{1}{x}$$

$$(x^\alpha)'=\alpha x^{\alpha-1}$$

$$(a^x)'=a^x\ln a$$

Pr.3: Vypočítajte deriváciu funkcie $f(x) = tg\,(3x+2) + 2x.e^x$

$$f'(x) = tg'\,(3x+2).(3x+2)' + (2x)'.e^x + 2x\,.(e^x)'$$

$$f\big(g(x)\big)' = f'(g(x)).g(x)' \qquad (ab)' = a'b + ab'$$

$$f'(x) = \frac{1}{\cos^2(3x+2)}.3 + 2.e^x + 2x.\,e^x$$

$$(tgx)' = \frac{1}{\cos^2 x} \qquad (3x+2)' = 3.1.x^0 + 0 = 3 \qquad (2x)' = 2.1.x^0 = 2 \qquad (e^x)' = e^x$$

$$f'(x) = \frac{1}{\cos^2(3x+2)}.3 + 2.e^x + 2x.\,e^x$$

Pr.4: 25 / 11: Vypočítajte deriváciu funkcie $f(x)=\sqrt{1+2\operatorname{tg}x}$

$$f(g(x))' = f'(g(x)) \cdot g'(x)$$

$$f(x) = \sqrt{1+2\operatorname{tg}x} = (1+2\operatorname{tg}x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(1+2\operatorname{tg}x)^{\frac{1}{2}-1} \cdot (1+2\operatorname{tg}x)'$$

$$(\operatorname{tg}x)' = \frac{1}{\cos^2 x}$$

$$f'(x) = \frac{1}{2}(1+2\operatorname{tg}x)^{-\frac{1}{2}} \cdot \left(0 + 2\frac{1}{\cos^2 x}\right)$$

$$x^{\alpha} = \alpha x^{\alpha-1}$$

$$f'(x) = \frac{1}{\cancel{2}} \frac{1}{\sqrt{1+2\operatorname{tg}x}} \cdot \cancel{2}\frac{1}{\cos^2 x}$$

$$f'(x) = \frac{1}{\sqrt{1+2\operatorname{tg}x}} \cdot \frac{1}{\cos^2 x}$$

Pr.5: Vypočítajte deriváciu funkcie $f(x) = \dfrac{x^2 + 2x + 3}{\operatorname{arctg} x} + \ln \sin 5x$

$$f'(x) = \frac{(2x + x).\operatorname{arctg} x - (x^2 + 2x + 3).\dfrac{1}{1 + x^2}}{(\operatorname{arctg} x)^2} + \frac{1}{\sin 5x}(\cos 5x)\,.5$$

$$\left(\frac{a}{b}\right)' = \frac{a'b - ab'}{b^2}$$

$$f\big(g(x)\big)' = f'(g(x)).\,g(x)'$$

$$f'(x) = \frac{(2x + x).\operatorname{arctg} x - (x^2 + 2x + 3).\dfrac{1}{1 + x^2}}{(\operatorname{arctg} x)^2} + 5\operatorname{cotg} 5x$$

Pr.6: Vypočítajte deriváciu funkcie $f(x) = \sin\sqrt{1+x^2} + \cos(\operatorname{tg} 3x)$

$$f'(x) = \cos\sqrt{1+x^2}\,.\,\frac{1}{2}(1+x^2)^{-\frac{1}{2}}.\,2x - \sin(\operatorname{tg} 3x)\frac{1}{\cos^2 3x}.\,3$$

$$f(g(x))' = f'(g(x)).\,g(x)'$$

$$f'(x) = \cos\sqrt{1+x^2}\,.\,\frac{x}{\sqrt{1+x^2}} - \sin(\operatorname{tg} 3x)\frac{1}{\cos^2 3x}.\,3$$

Pr.7: Vypočítajte deriváciu funkcie $f(x) = x.\arccos(x^2 + 5x) + \sqrt{2 - x^2 + 3^x}$

$$f'(x) = 1.\arccos(x^2 + 5x) + x\frac{-1}{\sqrt{1 - (x^2 + 5x)^2}}(2x + 5) + \frac{1}{2}(2 - x^2 + 3^x)^{-\frac{1}{2}}(-2x + 3^x \ln 3)$$

Testík: Vyberte správne tvrdenia.

1. Derivácia funkcie f(x) = $3x^2$ je
   a) 0,
   b) 3x,
   c) 6x.

2. Funkciu $y = \frac{1}{x}$ zderivujeme podľa vzťahu
   a) $(x^{\alpha})'$,
   b) (f . g)´,
   c) (f / g)´.

3. Funkciu $f(x) = \sqrt{2x}$ derivujeme ako
   a) zloženú funkciu,
   b) mocninovú funkciu x^{α},
   c) ako súčin dvoch funkcii (f . g)´.

4. Derivácia funkcie $y = x + \ln x$ je
   a) $y = 1 + \ln x$
   b) $y = x - \frac{1}{x}$
   c) $y = 1 + \frac{1}{x}$.