Matematika 1 – 5.cvičenie

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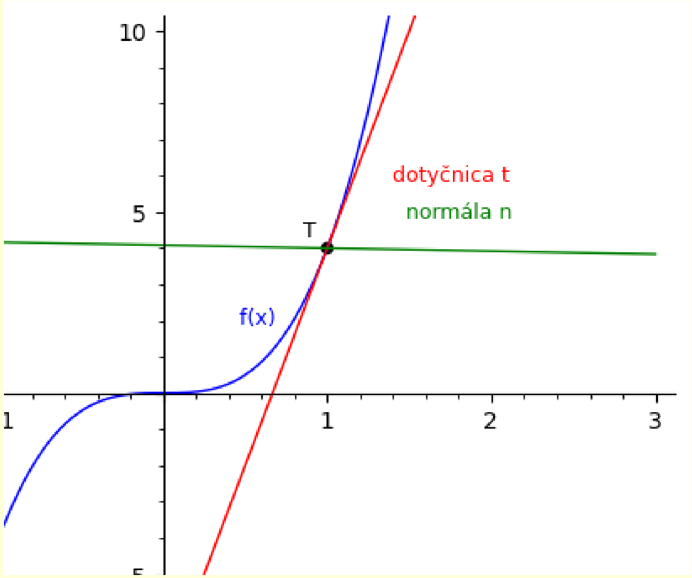
Geometrický význam derivácie funkcie



dotykový bod $T\,[x_0, y_0\,]$ ku grafu funkcie f(x)

Smernica dotyčnice = derivácia f ´(x_0) v dotykovom bode T ku grafu f (x)

$$k_t = f'(x_0)$$

Rovnica dotyčnice v dotykovom bode T ku grafu f (x)

$$t\!:\ \ y - y_0 = k_t\,(x - x_0)$$

Smernica normály $$k_n = -\frac{1}{f'(x_0)} = -\frac{1}{k_t}$$

Rovnica normály v dotykovom bode T ku grafu f (x) kolmo na dotyčnicu

$$n\!:\ \ y - y_0 = k_n\,(x - x_0)$$

Pr. 1 – 29 / 5 nájdite rovnicu dotyčnice a normály ku grafu funkcie $f(x)$ v dotykovom bode

$f(x)=\sqrt{2x}$ $\qquad T\left[\frac{1}{2},?\right]$

1. Dopočítať *y* – ovú súradnice dotykového bodu, pomocou *x* - ovej.

$$x_0 = \frac{1}{2}, \qquad y_0 = f(x_0) = \sqrt{2.\frac{1}{2}} = 1 \qquad T[\frac{1}{2}, 1]$$

2. Určiť prvú deriváciu funkcie $f´(x)$ a dopočítať smernicu dotyčnice k_t.

$$f´(x) = \frac{1}{2}(2x)^{\frac{-1}{2}}.2 = \frac{1}{\sqrt{2x}} \qquad k_t = f´(x_0) = \frac{1}{\sqrt{2x_0}} = \frac{1}{\sqrt{2.\frac{1}{2}}} = 1$$

3. Vypočítať smernicu normály k_n.

$$k_n = -\frac{1}{f´(x_0)} = -\frac{1}{k_t} = -\frac{1}{1} = -1$$

4. Vyjadriť rovnicu dotyčnice a normály.

t: $y - y_0 = k_t\,(x - x_0)$

$$y - 1 = 1\,(x - \frac{1}{2})$$

$$2y - 2 = 2x - 1$$

$$\underline{t\!: 0 = 2x - 2y + 1}$$

$\boldsymbol{n}\text{:}\ \boldsymbol{y - y_0 = k_n\,(x - x_0)}$

$$y - 1 = -1\left(x - \frac{1}{2}\right)$$

$$-2y + 2 = 2x - 1$$

$$0 = 2x + 2y - 3$$

$$\underline{n\text{: } 0 = 2x + 2y - 3}$$

Pr. 2 – 29 / 3

nájdite rovnicu dotyčnice a normály ku grafu funkcie $f(x)$ v dotykovom bode

$f(x)=x^3+9x+2$ $\qquad$ $T[0,?]$ $\qquad$ $t: 9x-y+2=0$

$n: x+9y-18=0$

$x_0 = 0,\qquad y_0 = f(x_0) = 0^3 + 9.0 + 2 = 2 \qquad T[0,2]$

$f'(x) = 3x^2 + 9 \qquad k_t = f'(x_0) = 3.0^2 + 9 = 9$

$$k_n = -\frac{1}{f'(x_0)} = -\frac{1}{k_t} = -\frac{1}{9}$$

$t:\ y - y_0 = k_t\,(x - x_0)$

$$y - 2 = 9\,(x - 0)$$

$$y - 2 = 9x$$

$$\underline{t:\ 0 = 9x - y + 2}$$

$n:\ y - y_0 = k_n\,(x - x_0)$

$$y - 2 = -\frac{1}{9}\,(x - 0)$$

$$-9y + 18 = x$$

$$\underline{n:\ 0 = x + 9y - 18}$$

Pr. 3 – 29 / 7

nájdite rovnicu dotyčnice a normály ku grafu funkcie $f(x)$ v dotykovom bode

$f(x) = 2x \ln x \qquad T[1,?]$

$x_0 = 1, \qquad y_0 = f(x_0) = 2.1.\ln 1 = 0 \qquad T[1,0] \qquad \boldsymbol{\ln 1 = 0}$

$f'(x) = 2.\ln x + 2x\frac{1}{x} = 2.\ln x + 2 \qquad \boldsymbol{k_t = f'(x_0) =} 2.\ln 1 + 2 = 2$

$$\boldsymbol{k_n = -\frac{1}{f'(x_0)} = -\frac{1}{k_t} = -\frac{1}{2}}$$

$\boldsymbol{t}$: $\boldsymbol{y - y_0 = k_t\,(x - x_0)}$

$y - 0 = 2\,(x - 1)$

$y = 2x - 2$

$\underline{t:\ 0 = 2x - y - 2}$

$\boldsymbol{n}$: $\boldsymbol{y - y_0 = k_n\,(x - x_0)}$

$y - 0 = -\frac{1}{2}\,(x - 1)$

$-2y = x - 1$

$\underline{n:\ 0 = x + 2y - 1}$

Pr. 4 – 29 / 9

nájdite rovnicu dotyčnice a normály ku grafu funkcie $f(x)$ v dotykovom bode

$f(x)=\frac{e^x}{2}+1$ $\quad T[0,?]$ $\quad t: x-2y+3=0$

$n: 4x+2y-3=0$

$$x_0 = 0, \qquad y_0 = f(x_0) = \frac{e^0}{2} + 1 = \frac{1}{2} + 1 = \frac{3}{2} \qquad T[0,\frac{3}{2}] \qquad e^0 = 1$$

$$f'(x) = \frac{e^x}{2} \qquad k_t = f'(x_0) = \frac{e^0}{2} = \frac{1}{2}$$

$$k_n = -\frac{1}{f'(x_0)} = -\frac{1}{k_t} = -2$$

$$t: \; y - y_0 = k_t\,(x - x_0) \qquad y - \frac{3}{2} = \frac{1}{2}(x-0)$$

$$2y - 3 = x$$

$$\underline{t: \; 0 = x - 2y + 3}$$

$$n: \; y - y_0 = k_n\,(x - x_0) \qquad y - \frac{3}{2} = -2\,(x-0)$$

$$-2y + 3 = 4x$$

$$\underline{n: \; 0 = 4x + 2y - 3}$$

Dú – str. 29 / 4, 6, 8, 10

L ´ Hospitalovo pravidlo

Pre výpočet limít, kde sa použije derivácia. Platí za určitých podmienok

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

1. limity s neurčitosťou $\frac{\infty}{\infty}$, $\frac{0}{0}$ - priamo použijeme toto pravidlo

Pr. 1 – 34 / 8

$$\lim_{x \to 0} \frac{\cos x - \cos^3 x}{x^2} \overset{\frac{0}{0}}{=} \lim_{x \to 0} \frac{\cos x - \cos^3 x}{x^2} \overset{L´H}{=} \lim_{x \to 0} \frac{(\cos x - \cos^3 x)´}{(x^2)´} =$$

$$= \lim_{x \to 0} \frac{-\sin x + 3\cos^2 x . \sin x}{2x} \overset{\frac{0}{0}}{=}$$

$$\overset{L´H}{=} \lim_{x \to 0} \frac{-\cos x - 3.2. \cos x . \sin x . \sin x + 3\cos^2 x . \cos x}{2} =$$

$$= \frac{-\cos 0 - 6. \cos 0 . \sin 0 . \sin 0 + 3\cos^2 0 . \cos 0}{2} = \frac{-1 - 0 + 3.1}{2} = 1$$

Pr. 2 – 34 / 5

$$\lim_{x\to 0}\frac{6^x-3^x}{x} \qquad \ln 2$$

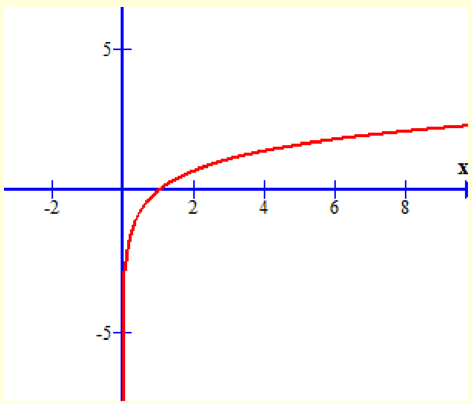
$$\lim_{x\to 0}\frac{6^x-3^x}{x} \overset{\frac{0}{0}}{=} \overset{L'H}{\lim_{x\to 0}}\frac{(6^x-3^x)'}{x'} = \lim_{x\to 0}\frac{6^x\ln 6-3^x\ln 3}{1} = 6^0\ln 6-3^0\ln 3=\ln\frac{6}{3}=\ln 2$$

$$\ln a-\ln b=\ln\frac{a}{b}$$

Pr. 3 – 35 / 13

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} \overset{\frac{\infty}{\infty}\ L'H}{=} \lim_{x\to\infty}\frac{(\ln x)'}{(\sqrt{x})'} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2}x^{\frac{-1}{2}}} = \lim_{x\to\infty}\frac{2\sqrt{x}}{x} \overset{\frac{\infty}{\infty}\ L'H}{=} \lim_{x\to\infty}\frac{(2\sqrt{x})'}{x'} \lim_{x\to\infty}\frac{2\,\frac{1}{2}\,x^{\frac{-1}{2}}}{1} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0$$



Dú – str. 34,35 / 3, 4, 6, 7, 9, 11, 12

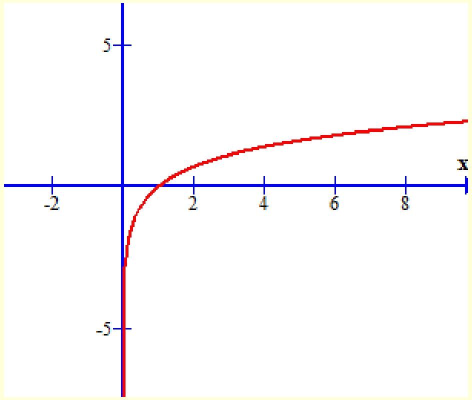
2. limity s neurčitosťou ∞ – ∞, upravíme na spoločného menovateľa, dostaneme neurčitosť $\frac{\infty}{\infty}$ alebo $\frac{0}{0}$ a potom použijeme toto pravidlo

Pr. 4 – 35 / 16

$$\lim_{x\to 1+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) \overset{\infty - \infty}{=} \lim_{x\to 1^+}\frac{x.\ln x-(x-1)}{(x-1).\ln x} \overset{\frac{0}{0}}{=} \overset{L'H}{} \lim_{x\to 1^+}\frac{[x.\ln x-(x-1)]'}{[(x-1).\ln x]'} =$$

$$\lim_{x\to 1^+}\frac{1.\ln x + x\frac{1}{x} - 1}{1.\ln x + (x-1)\frac{1}{x}} = \lim_{x\to 1^+}\frac{\ln x}{\ln x + (1-\frac{1}{x})} \overset{\frac{0}{0}}{=} \overset{L'H}{} \lim_{x\to 1^+}\frac{(\ln x)'}{\left[\ln x + (1-\frac{1}{x})\right]'} =$$

$$\lim_{x\to 1^+}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{1+1} = \frac{1}{2}$$



Pr. 5 – 35 / 19

$$\lim_{x\to 0+}\left(\frac{1}{2x}-\frac{1}{\sin x}\right) \qquad -\infty$$

$$\lim_{x\to 0+}\left(\frac{1}{2x}-\frac{1}{\sin x}\right) \overset{\infty - \infty}{=} \lim_{x\to 0^+}\frac{\sin x - 2x}{2x.\sin x} \overset{\frac{0}{0}}{=} \overset{L'H}{\lim_{x\to 0^+}}\frac{[\sin x - 2x]'}{[2x.\sin x]'} =$$

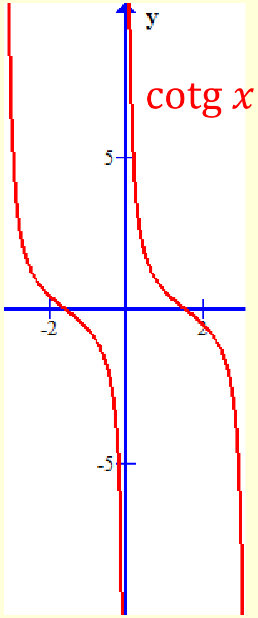
$$\lim_{x\to 0^+}\frac{\cos x \ - 2}{2.\sin x + 2x.\cos x} = \frac{\cos 0^+ - 2}{2.\sin 0^+ + 2.0^+.\cos 0^+} = \frac{1-2}{0^+ + 0^+} = \frac{-1}{0^+} = -\infty$$

Dú – str. 34,35 / 17,18, 20

3. limity s neurčitosťou $0.\infty$, upravíme limitu na tvar ,

$$\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}} \quad \textbf{alebo} \quad \lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$$

dostaneme neurčitosť $\frac{\infty}{\infty}$ alebo $\frac{0}{0}$ a potom použijeme toto pravidlo

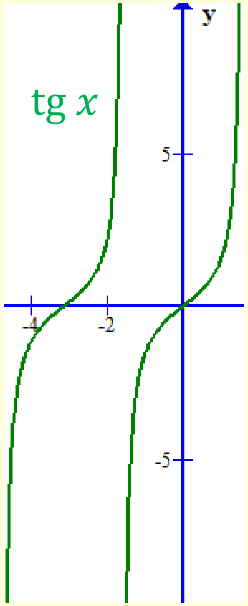


Pr. 6

$$= \lim_{x\to 0^+} (1-\cos x).\operatorname{cotg} x \overset{0.\infty}{=} \lim_{x\to 0^+} \frac{1-\cos x}{\frac{1}{\operatorname{cotg} x}} = \lim_{x\to 0^+} \frac{1-\cos x}{\operatorname{tg} x} \overset{\frac{0}{0}}{=}$$

L´H

$$\lim_{x\to 0^+} \frac{(1-\cos x)'}{(\operatorname{tg} x)'} = \lim_{x\to 0^+} \frac{\sin x}{\frac{1}{\cos^2 x}} = \lim_{x\to 0^+} \sin x.\cos^2 x = 0.1 = 0$$

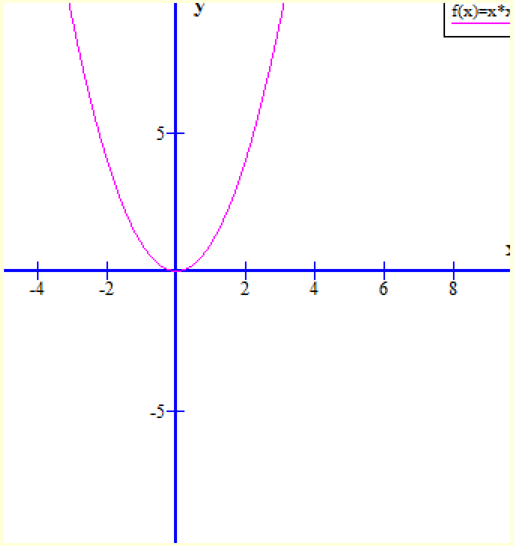


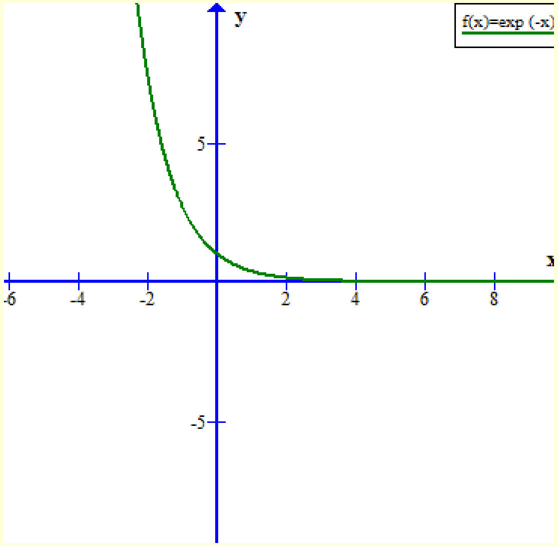
Pr. 7 – 35 / 21

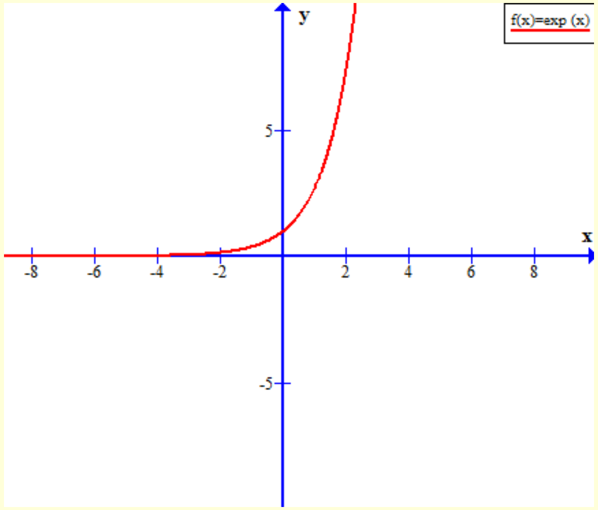
$$\lim_{x\to\infty} x^2 e^{-x} \qquad 0$$

$$= \lim_{x\to\infty} x^2 e^{-x} \overset{\infty.0}{=} \lim_{x\to\infty} \frac{x^2}{\frac{1}{e^{-x}}} \overset{\frac{\infty}{\infty}}{=} \overset{L'H}{\lim_{x\to\infty}} \frac{(x^2)'}{(e^x)'} = \lim_{x\to\infty} \frac{2x}{e^x} \overset{\frac{\infty}{\infty}}{=} \overset{L'H}{\lim_{x\to\infty}} \frac{(2x)'}{(e^x)'} =$$

$$= \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$







Dú – str. 34,35 / 22

3. Malá písomka

Skupina A: Vypočítajte limitu funkcie $\lim_{x \to -1} \frac{x(x+1)}{x^2+3x+2}$

Skupina B: Vypočítajte limitu funkcie $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$

Skupina E: (online) Vypočítajte limitu funkcie $\lim_{x \to 0} \frac{\sin 3x}{\operatorname{tg} x}$

3. Malá písomka

Skupina C: Vypočítajte limitu funkcie $\lim_{x \to 0} \frac{\operatorname{tg} 8x}{8}$

Skupina D: Vypočítajte limitu funkcie $\lim_{x \to \infty} (\sqrt{x+2} - \sqrt{x})$

Skupina F: (online) Vypočítajte limitu funkcie $\lim_{x \to \infty} \frac{x^4 - 3x^2 + 1}{2x^5 + 10x}$

Skupina A

Vypočítajte limitu funkcie $\lim_{x \to -1} \frac{x(x+1)}{x^2+3x+2}$

0,2 b 0,2 b

$$\lim_{x \to -1} \frac{x^2+x}{x^2+3x+2} \overset{\frac{0}{0}}{=} \lim_{x \to -1} \frac{x(x+1)}{(x+1)(x+2)} = \lim_{x \to -1} \frac{x}{(x+2)} = \frac{-1}{-1+2} = -1$$

0,2 b 0,2 b 0,1 b 0,1 b

Skupina B

Vypočítajte limitu funkcie $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$

0,1 b 0,2 b 0,2 b 0,1 b

$$\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} \overset{\frac{0}{0}}{=} \lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x \to 4} \frac{(\sqrt{x})^2-2^2}{(x-4)(\sqrt{x}+2)} = \lim_{x \to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} =$$

$$\lim_{x \to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

0,2 b 0,2 b

Skupina C Vypočítajte limitu funkcie $\lim_{x\to 0} \frac{\operatorname{tg} 8x}{x}$

0,2 b

$$\lim_{x\to 0} \frac{\operatorname{tg} 8x}{8} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\sin 8x}{8\cos 8x} = \lim_{x\to 0} \frac{\sin 8x}{8\cos 8x}\frac{x}{x} = \lim_{x\to 0} \frac{\sin 8x}{8x} \lim_{x\to 0} \frac{x}{\cos 8x} = 1.\frac{0}{\cos 0} = 1.\frac{0}{1} = 0$$

0,2 b 0,2 b 0,2 b 0,1 b 0,1 b

Alebo $\lim_{x\to 0} \frac{\operatorname{tg} 8x}{8} = \frac{0}{8} = 0$

Skupina D Vypočítajte limitu funkcie $\lim_{x\to\infty} (\sqrt{x+2} - \sqrt{x})$

0,1 b

$$\lim_{x\to\infty} (\sqrt{x+2} - \sqrt{x}) \overset{\infty-\infty}{=} \lim_{x\to\infty} (\sqrt{x+2} - \sqrt{x}).\frac{\sqrt{x+2} + \sqrt{x}}{\sqrt{x+2} + \sqrt{x}} = \lim_{x\to\infty} \frac{(\sqrt{x+2})^2 - \sqrt{x}^2}{\sqrt{x+2} + \sqrt{x}} =$$

0,2 b 0,2 b

$$\lim_{x\to\infty} \frac{x+2-x}{\sqrt{x+2} + \sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x+2} + \sqrt{x}} = = \frac{2}{\infty} = 0$$

0,2 b 0,1 b 0,2 b

Skupina E: (online) Vypočítajte limitu funkcie $\lim_{x \to 0} \frac{\sin 3x}{\operatorname{tg} x}$

Skupina F: (online) Vypočítajte limitu funkcie $\lim_{x \to \infty} \frac{x^4 - 3x^2 + 1}{2x^5 + 10x}$

0,1 b 0,1 b 0,2 b 0,1 b

$$\lim_{x \to 0} \frac{\sin 3x}{\operatorname{tg} x} \overset{\frac{0}{0}}{=} \lim_{x \to 0} \frac{\sin 3x}{\frac{\sin x}{\cos x}} = \lim_{x \to 0} \frac{\sin 3x \cos x}{\sin x} \frac{3x}{3x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \lim_{x \to 0} \frac{x}{\sin x} \lim_{x \to 0} 3.\cos x = 1.1.3.\cos 0 = 3$$

0,1 b 0,2 b 0,1 b 0,1 b

$$\lim_{x \to \infty} \frac{x^4 - 3x^2 + 1}{2x^5 + 10x} \overset{\frac{\infty}{\infty}}{=} \lim_{x \to \infty} \frac{\frac{x^4}{x^5} - \frac{3x^2}{x^5} + \frac{1}{x^5}}{\frac{2x^5}{x^5} + \frac{10x}{x^5}} = \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{3}{x^3} + \frac{1}{x^5}}{2 + \frac{10}{x^4}} = \frac{\frac{1}{\infty} - \frac{3}{\infty} + \frac{1}{\infty}}{2 + \frac{10}{\infty}} = \frac{0}{2 + 0} = 0$$