Matematika 1 – 5.cvičenie

RNDr. Z. Gibová, PhD.

Limita funkcie

Zápis limity: $\lim\limits_{x \to a} f(x) = \mathrm{b}$

Pri výpočte limity:

a) dosadíme za neznáme x číslo *a*, ak **žiadna neurčitosť**, vypočítame limitu

Pr. 1 – 16 / 1

Typy neurčitosti: $\frac{\mathbf{0}}{\mathbf{0}}, \frac{\infty}{\infty}, \infty - \infty, \mathbf{1}^{\infty}$

$$\lim_{x \to 2} \frac{x^2 + 5}{x^2 - 3}$$

Pr. 2

$$\lim_{x \to 1} \frac{x^3 + 3x + 5}{x^2 - 4} = \frac{1^3 + 3.1 + 5}{1^2 - 4} = \frac{9}{-3} = -3$$

b) dosadíme za neznáme x číslo *a*, ak **neurčitosť** $\mathbf{\frac{0}{0}}$ **,** urobíme rozklad na súčin v čitateli aj v menovateli (ak v čitateli a menovateli polynómy)

Pr. 3 – 16 / 7

$$\lim_{x \to -1} \frac{x^2 - 1}{x^2 + 6x + 5}$$

Pr. 4 – 16 / 3

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \qquad 4$$

$$\lim_{x \to 2} \frac{(2)^2 - 4}{2 - 2} = \frac{0}{0}$$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \overset{\frac{0}{0}}{=} \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim_{x \to 2} x + 2 = 2 + 2 = 4$$

Dú : 16 / 4, 8

c) dosadíme za neznáme x číslo *a*, ak **neurčitosť** $\frac{\mathbf{0}}{\mathbf{0}}$, rozšírime vhodnou jednotkou (ak v čitateli alebo menovateli druhá odmocnina s použitím vzorca $a^2 - b^2$)

Pr. 5 – 17 / 17

$$\lim_{x \to 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$$

$$(\boldsymbol{a} - \boldsymbol{b})(\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a}^2 - \boldsymbol{b}^2$$

Pr. 6 – 16 / 14

$$\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x} \qquad 0$$

$$\lim_{x\to 0}\frac{\sqrt{1+0^2}-1}{0}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\overbrace{\sqrt{1+x^2}}^{a}-\overbrace{1}^{b}}{x}.\underbrace{\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1}}_{=1} = \lim_{x\to 0}\frac{\overbrace{1+x^2}^{a^2}-\overbrace{1}^{b^2}}{x.(\sqrt{1+x^2}+1)} =$$

$$\lim_{x\to 0}\frac{x}{(\sqrt{1+x^2}+1)} = \frac{0}{(\sqrt{1+0^2}+1)} = 0$$

$$(\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{b})=\boldsymbol{a}^2-\boldsymbol{b}^2$$

Dú : 16 / 11,12

d) goniometrická funkcia, dosadíme, ak **neurčitosť** $\mathbf{\frac{0}{0}}$, použijeme vzorec

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{\otimes \to 0} \frac{\sin \otimes}{\otimes} = 1$$

Pr. 7 – 17 / 22

$$\lim_{x \to 0} \frac{\operatorname{tg} 2x}{\sin 5x} \qquad \frac{2}{5}$$

Pr. 8 – 17 / 19

$$\lim_{x \to 0} \frac{tg\, 5.0}{0} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{\operatorname{tg} 5x}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\sin 5x}{x.\cos 5x} = \lim_{x\to 0} \frac{\sin 5x}{x.\cos 5x}\,\frac{5}{5} = \lim_{x\to 0} \frac{\sin 5x}{5\,x}\;\frac{5}{\cos 5x} =$$

$$\lim_{x\to 0} \frac{\sin 5x}{5\,x} . \lim_{x\to 0} \frac{5}{\cos 5x} = 1.\frac{5}{\cos 0} = 5$$

Dú : 17 / 20, 21

e) dosadíme, ak **neurčitosť** $\frac{\infty}{\infty}$, delíme čitateľa aj menovateľa najvyššou mocninou v menovateli (ak v čitateli a menovateli polynómy)

Platí: $\infty . \infty = \infty, \quad -\infty . -\infty = \infty, \quad -\infty . \infty = -\infty, \quad c . \infty = \infty$

$\infty + \infty = \infty, \quad -\infty - \infty = -\infty, \quad \infty + c = \infty, \quad \sqrt{\infty} = \infty$

$$\lim_{x \to \infty} \frac{1}{x} = 0, \lim_{x \to \infty} \frac{1}{x^n} = 0, \lim_{x \to \infty} x = \infty$$

Pr. 9 – 17 / 29

$$\lim_{x \to \infty} \frac{x^3 + x}{x^4 - 3x^2 + 1}$$

Pr. 10 – 17 / 30

$$\lim_{x\to\infty}\frac{2x^4-5x}{10x^2-3x+1} \qquad \infty$$

$$\lim_{x\to\infty}\frac{\infty}{\infty+1}=\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\frac{2x^4-5x}{10x^2-3x+1}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{\frac{2x^4}{x^2}-\frac{5x}{x^2}}{\frac{10x^2}{x^2}-\frac{3x}{x^2}+\frac{1}{x^2}}=\lim_{x\to\infty}\frac{2x^2-\frac{5}{x}}{10-\frac{3}{x}+\frac{1}{x^2}}=\frac{\infty-\frac{1}{\infty}}{10-\frac{3}{\infty}+\frac{1}{\infty}}=\infty$$

=0

najvyššia mocnina, delíme x^2

$\lim_{x\to\infty}\frac{4x^2-5x+2}{2x^2+3}$ Dú 2

$\lim_{x\to\infty}\frac{2x+5}{3x-1}$ Dú $\frac{2}{3}$

g) dosadíme, ak **neurčitosť** $\infty - \infty$, rozšírime vhodnou jednotkou a potom delíme čitateľa aj menovateľa najvyššou mocninou v menovateli

Pr. 11

$$\lim_{x\to\infty}\left(\sqrt{x^2+5x+6}-\sqrt{x^2-3}\right)$$

Pr. 12 – 17 / 34

$$\lim_{x\to\infty}\left(\sqrt{x-2}-\sqrt{x}\right) \qquad 0$$

$$\lim_{x\to\infty} \sqrt{\infty}-\sqrt{\infty}=\infty-\infty$$

$$\lim_{x\to\infty}\left(\sqrt{x-2}-\sqrt{x}\right)\overset{\infty-\infty}{=}\lim_{x\to\infty}\left(\sqrt{x-2}-\sqrt{x}\right)\frac{\sqrt{x-2}+\sqrt{x}}{\sqrt{x-2}+\sqrt{x}}=\lim_{x\to\infty}\frac{x-2-x}{\sqrt{x-2}+\sqrt{x}}=$$

$$\lim_{x\to\infty}\frac{-\frac{2}{\sqrt{x}}}{\sqrt{\frac{x}{x}-\frac{2}{x}}+\sqrt{\frac{x}{x}}}=\lim_{x\to\infty}\frac{0}{\sqrt{1-0}+\sqrt{1}}=0$$

najvyššia mocnina je $\sqrt{x}$

Dú : 17 / 35, 36

h) dosadíme, ak **neurčitosť $\mathbf{1}^{\infty}$**, použijeme jeden zo vzorcov

$$\lim_{x\to\infty}\left(1\pm\frac{1}{x}\right)^{x}=e^{\pm 1}$$

$$\lim_{x\to\infty}\left(1+\frac{k}{f(x)}\right)^{f(x)}=e^{k}\ ,\qquad f(x)\to\infty$$

Pr. 13

$$\left(1+\frac{2}{\infty}\right)^{\infty} = \mathbf{1}^{\infty}$$

$$\lim_{x\to\infty}\left(1+\frac{3}{x-5}\right)^{2x} \overset{\mathbf{1}^{\infty}}{=} \lim_{x\to\infty}\left(1+\frac{3}{x-5}\right)^{2x\frac{(x-5)}{x-5}} = \lim_{x\to\infty}\left[\left(1+\frac{3}{x-5}\right)^{x-5}\right]^{\frac{2x}{x-5}} =$$

$$e^{3.\lim\limits_{x\to\infty}\frac{2x}{x-5}} = e^{3.\frac{2}{1-0}} = e^{6}$$

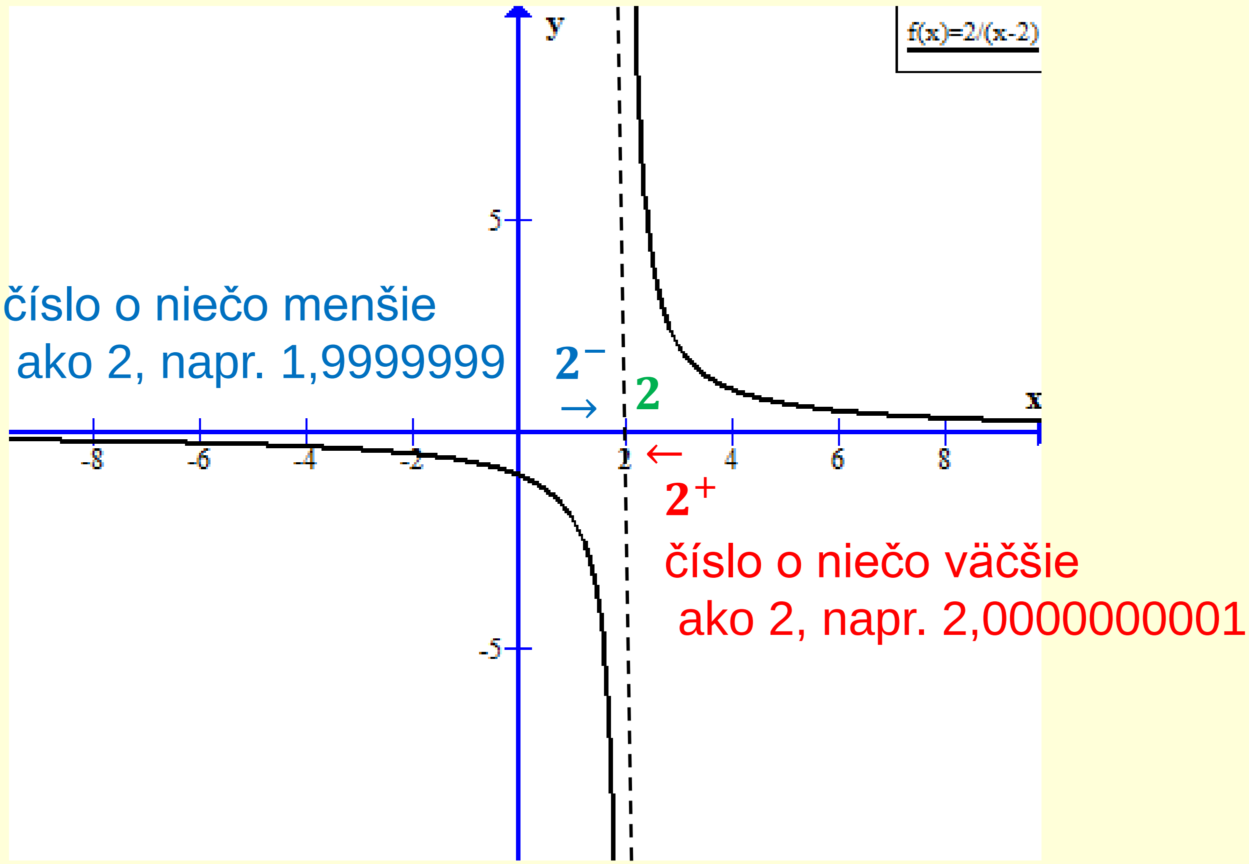
$$\lim_{x\to\infty}\left(1+\frac{3}{x+1}\right)^{x-1}$$ Dú e^{3}

Asymptoty funkcie

Jednostranné limity:

limita pre *x* idúce k ***a*** sprava $\lim_{x \to a^+} f(x)$

limita pre *x* idúce k ***a*** zľava $\lim_{x \to a^-} f(x)$

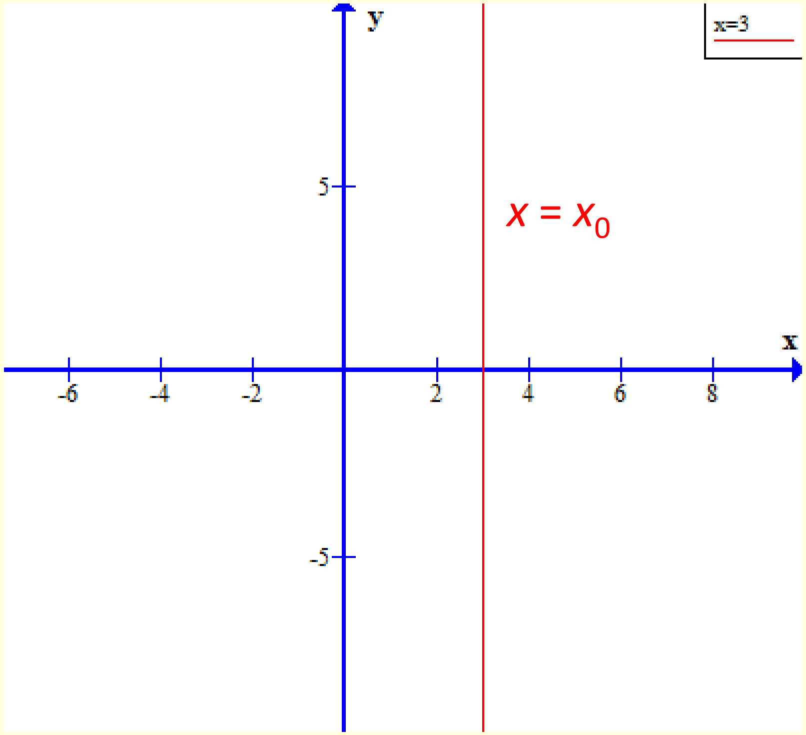


Asymptota je priamka, ktorá opisuje správanie sa krivky.

Asymptota bez smernice (ABS): vypočítame jednostranné limity v bode nespojitosti x_0 (bod v ktorom funkcie nie je definovaná), ak vyjdú nevlastné čísla $\pm\infty$ (stačí jedna z limít rovná $\pm\infty$, potom funkcia má ABS)

$$\lim_{x \to x_0^+} f(x) = \pm\infty \qquad \lim_{x \to x_0^-} f(x) = \pm\infty$$

potom ABS je $x = x_0$

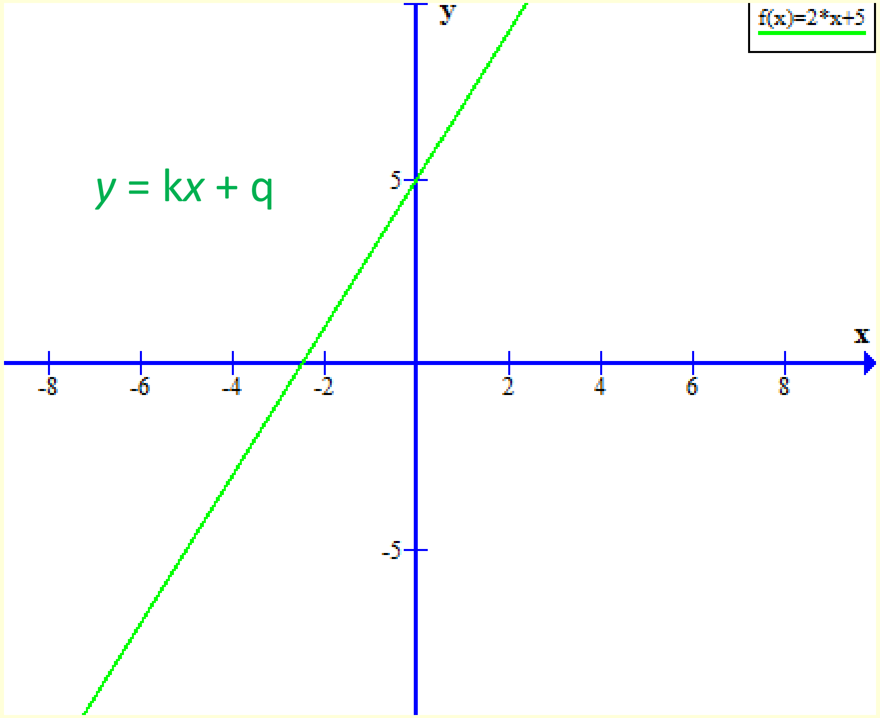


Asymptota so smernicou (ASS): priamka $y = kx + q$

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} \qquad\qquad q = \lim_{x \to \pm\infty} [f(x) - kx]$$

vypočítame k, q, ak vyjdú vlastné čísla, potom

ASS je $y = k_1 x + q_1$, $y = k_2 x + q_2$



Pr.1: Určte ASS a ABS funkcie

$$y = \frac{3(x-1)^2}{x-2}$$

1. Určiť D(f) a bod nespojitosti.

$$x - 2 \neq 0$$

$$x \neq 2\,,\quad D(f) = R - \{2\}$$

bod nespojitosti $x_0 = 2$

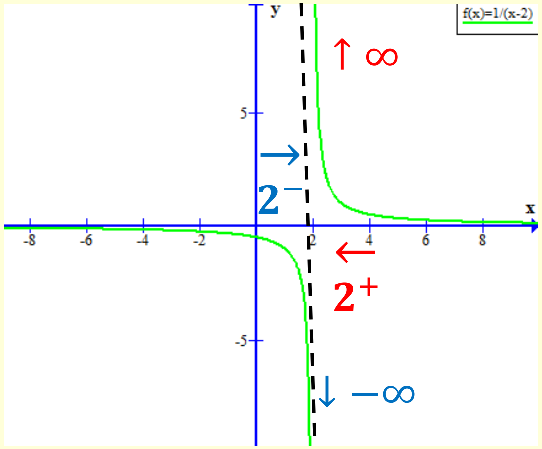
2. Určiť ABS v bode nespojitosti

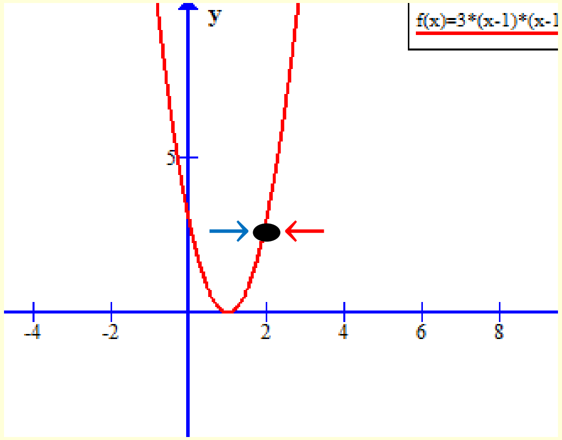
$$\frac{1}{2^+ - 2} = \frac{1}{0^+} \qquad 3(2^+ - 1)^2 = +\,\check{c}\acute{\imath}slo$$

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to 2^+} \frac{3(x-1)^2}{x-2} = \lim_{x\to 2^+} \frac{1}{x-2} \cdot \lim_{x\to 2^+} 3(x-1)^2 = +\infty$$

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to 2^-} \frac{3(x-1)^2}{x-2} = \lim_{x\to 2^-} \frac{1}{x-2} \cdot \lim_{x\to 2^-} 3(x-1)^2 = -\infty$$

$$\frac{1}{2^- - 2} = \frac{1}{0^-} \qquad 3(2^- - 1)^2 = +\,\check{c}\acute{\imath}slo$$





ABS: $x = 2$

3. Určiť ASS

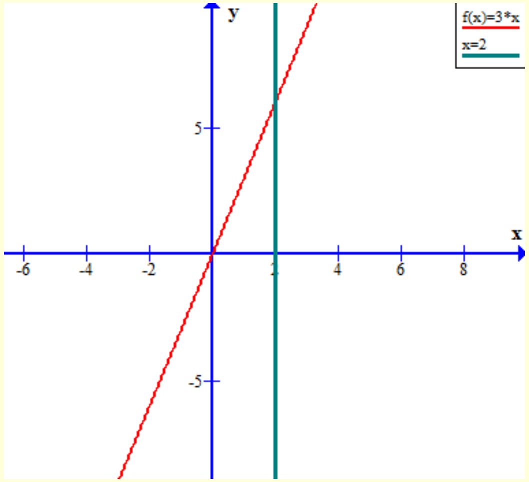
$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{3(x-1)^2}{x(x-2)}=\lim_{x\to\infty}\frac{3x^2-6x+3}{x^2-2x}=3$$

$$k=\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{3(x-1)^2}{x(x-2)}=\lim_{x\to-\infty}\frac{3x^2-6x+3}{x^2-2x}=3$$

$$q=\lim_{x\to\infty}[f(x)-kx]=\lim_{x\to\infty}\left[\frac{3(x-1)^2}{(x-2)}-3x\right]=\lim_{x\to\infty}\left[\frac{3x^2-6x+3-3x^2+6x}{(x-2)}\right]$$

$$=\lim_{x\to\infty}\left[\frac{3}{(x-2)}\right]=0 \qquad q=\lim_{x\to-\infty}\left[\frac{3}{(x-2)}\right]=0$$

ASS: $y=3x \quad pre\ \ x\to\pm\infty$



ASS: $y=3x \qquad pre\ \ x\to\pm\infty$

ABS: $x=2$

Pr.2: 47/ 2 Určte ASS a ABS funkcie

$$y = x^3 + 3x^2 - 2$$