integrál typu $\int \frac{1}{\sqrt{ax^2+bx+c}}dx \quad ; a \neq 0$

↓ úprava výrazu pod odmocninou na štvorec

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+c \qquad \int \frac{dx}{\sqrt{x^2+k}} = \ln\left|x+\sqrt{x^2+k}\right|+c$$

príklad ① $\int \frac{1}{\sqrt{2x^2-4x+5}}dx = \int \frac{1}{\sqrt{2\cdot\left(x^2-2x+\frac{5}{2}\right)}}dx = \frac{1}{\sqrt2}\int\frac{dx}{\sqrt{x^2-2x+\frac{5}{2}}} = \frac{1}{\sqrt2}\int\frac{dx}{\sqrt{(x-1)^2-1+\frac{5}{2}}} =$

$(a-b)^2$; x^2-2x+1

$= \frac{1}{\sqrt2}\int\frac{dx}{\sqrt{(x-1)^2+\frac{3}{2}}} = \left|\begin{matrix} x-1=t \\ dx=dt\end{matrix}\right| = \frac{1}{\sqrt2}\int\frac{dt}{\sqrt{t^2+\frac32}} = \frac{1}{\sqrt2}\ln\left|t+\sqrt{t^2+\frac32}\right|+c =$

$= \frac{1}{\sqrt2}\ln\left|x-1+\sqrt{(x-1)^2+\frac32}\right|+c$

podobne sa integrovala funkcia $\int\frac{1}{ax^2+bx+c}dx$ (úprava na □).

② $\int\frac{1}{\sqrt{3+2x-x^2}}dx = \int\frac{1}{\sqrt{-(x^2-2x-3)}}dx = \int\frac{1}{\sqrt{-[(x-1)^2-1-3]}}dx = \int\frac{1}{\sqrt{-[(x-1)^2-4]}}dx =$

x^2-2x+1

$= \int\frac{1}{\sqrt{4-(x-1)^2}}dx = \left|\begin{matrix} x-1=t \\ dx=dt\end{matrix}\right| = \int\frac{1}{\sqrt{4-t^2}}dt = \arcsin\frac{t}{2}+c = \arcsin\frac{x-1}{2}+c$ (4 = a^2)

INTEGRÁL TYPU $\int\frac{P_m(x)}{\sqrt{ax^2+bx+c}}dx = Q_{m-1}(x)\cdot\sqrt{ax^2+bx+c} + k\int\frac{1}{\sqrt{ax^2+bx+c}}dx$

(vypočítať, vypočítať, vypočítať)

metóda neurčitých koeficientov.
Výhoda: dáva nám výsledok.

použitie metódy na príklade:

$$\int\frac{3+2x-x^2}{\sqrt{3+2x-x^2}}dx = (Ax+B)\cdot\sqrt{3+2x-x^2} + k\int\frac{1}{\sqrt{3+2x-x^2}}dx$$

($P_2(x)$) zderivujeme

teď určím A, B, k a vypočítam $\int\frac{1}{\sqrt{3+2x-x^2}}dx = \arcsin\frac{x-1}{2}+c$

$$\frac{3+2x-x^2}{\sqrt{3+2x-x^2}} = A\cdot\sqrt{3+2x-x^2} + (Ax+B)\cdot\frac{1}{2}(3+2x-x^2)^{-\frac12}\cdot(2-2x) + k\frac{1}{\sqrt{3+2x-x^2}} \quad \Big/ \cdot\sqrt{3+2x-x^2}$$

$3+2x-x^2 = A(3+2x-x^2) + (Ax+B)\cdot(1-x) + k$

polynóm = polynóm

$3+2x-x^2 = 3A + 2Ax - Ax^2 + Ax - Ax^2 + B - Bx + k$

$x^2: \ -1 = -2A \Rightarrow A = \frac12$

$x: \ 2 = 2A + A - B$
$2 = 3A - B$
$2 = \frac32 - B$
$B = -\frac12$

$x^0: \ 3 = 3A + B + k \Rightarrow k = 2$

výsledok: $\int\frac{3+2x-x^2}{\sqrt{3+2x-x^2}}dx = \left(\frac12 x - \frac12\right)\cdot\sqrt{3+2x-x^2} + 2\cdot\arcsin\frac{x-1}{2}+c$

INTEGROVANIE GONIOMETRICKÝCH FUNKCIÍ

vieme: vzorec
$\int \sin x\,dx = -\cos x + c$
$\int \cos x\,dx = \sin x + c$

$\int \operatorname{tg} x\,dx = -\int\frac{-\sin x}{\cos x}dx = -\ln|\cos x|+c$ (f', f)

$\int \operatorname{cotg} x\,dx = \int\frac{\cos x}{\sin x}dx = \ln|\sin x|+c$

(I) pri integrovaní často využívame vzťahy ktoré platia medzi goniom. funkciami

$\sin^2x + \cos^2x = 1$
$\sin 2x = 2\sin x\cdot\cos x$
$\cos 2x = \cos^2x - \sin^2x$

! $\sin^2 x = \frac{1-\cos 2x}{2}$; $\cos^2 x = \frac{1+\cos 2x}{2}$ používame pri integrovaní $\int\cos^{2k}x\,dx$; $\int\sin^{2k}x\,dx$

príklady: ① $\int\sin x\cdot\cos x\,dx = \frac12\int 2\sin x\cdot\cos x\,dx = \frac12\int\sin 2x\,dx = \left|\begin{matrix}2x=t\\2dx=dt\\dx=\frac{dt}{2}\end{matrix}\right| =$

$= \frac12\int\sin t\,\frac{dt}{2} = \frac14(-\cos t)+c = -\frac14\cos 2x+c$

② $\int\cos^2x\,dx = \int\frac{1+\cos 2x}{2}dx = \frac12\int(1+\cos 2x)dx = \frac12\left(x+\frac{\sin 2x}{2}\right)+c$

③ $\int\sin^2x\cdot\cos^2x\,dx = \int\frac{1-\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}dx = \frac14\int(1-\cos 2x)\cdot(1+\cos 2x)dx =$ $(a-b)\cdot(a+b)$

$= \frac14\int(1-\cos^2 2x)dx = \frac14\int\left(1-\frac{1+\cos 4x}{2}\right)dx = \frac14\int\left(1-\frac12-\frac{\cos 4x}{2}\right)dx =$

$= \frac14\int\left(\frac12-\frac{\cos 4x}{2}\right)dx = \frac18\int(1-\cos 4x)dx = \frac18\left(x-\frac{\sin 4x}{4}\right)+k$

iný spôsob: $\int(\sin x\cdot\cos x)^2dx = \int\left(\frac{\sin 2x}{2}\right)^2dx = \frac14\int\sin^2 2x\,dx =$

$\sin x\cdot\cos x = \frac{\sin 2x}{2}$

TEÓRIA na 1. ZP prednášky 1. až 6. týždeň + komplexné čísla (7. týždeň už nie) ↓ určitý integrál prof. Dzurina

(II) integrály typu: $\int R[\sin x]\cdot\cos x\,dx$, $\sin x = t$

$\int R[\cos x]\cdot\sin x\,dx$; subst. $\cos x = t$

príklad: ① $\int(2+\cos^2x-4\cos x)\cdot\sin x\,dx = \left|\begin{matrix}\text{sub. } \cos x=t\\-\sin x\,dx=dt\\\sin x\,dx=-dt\end{matrix}\right| = \int(2+t^2-4t)\cdot(-dt) =$ ($R[\cos x]$, $\cdot\sin x\,dx$)

$= \int(-2-t^2+4t)dt = -2t-\frac{t^3}{3}+4\frac{t^2}{2}+c = -2\cos x-\frac{\cos^3x}{3}+2\cos^2x+c$

② $\int\frac{\sin^3x}{2+\cos x}dx = \left\{\int\left(\text{same } \cos x\right)\cdot\sin x\,dx\right\} = \int\frac{\sin^2x\,\sin x}{2+\cos x}dx = \int\frac{1-\cos^2x}{2+\cos x}\cdot\sin x\,dx$

$= \left|\begin{matrix}\cos x=t\\-\sin x\,dx=dt\\\sin x\,dx=-dt\end{matrix}\right| = -\int\frac{1-t^2}{2+t}dt = \int\frac{t^2-1}{2+t}dt = \int\left(t-2+\frac{3}{t+2}\right)dt$

RACIONÁLNA FUNKCIA :) (rýdzorac. je to parc. zlomok)

$(t^2-1):(t+2) = t-2$
$-(t^2+2t)$
$-2t-1$
$-(-2t-4)$
3

$= \frac{t^2}{2}-2t+3\ln|t+2|+c =$

$= \frac{\cos^2x}{2}-2\cos x+3\ln|\cos x+2|+c$

(III) univerzálna substitúcia pre integrál $\int R[\sin x, \cos x]dx$

$\operatorname{tg}\frac{x}{2} = t$

$\frac{x}{2} = \operatorname{arctg} t$

$x = 2\operatorname{arctg} t$

$dx = \frac{2}{1+t^2}dt$

$\sin x = \frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \Big| \cdot\frac{1}{\cos^2\frac{x}{2}} = \frac{2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}+1} = \frac{2t}{t^2+1}$ ($=1$; $\operatorname{tg}\frac{x}{2}$)

$\cos x = \frac{1-t^2}{1+t^2}$

↑ táto substitúcia prevedie ľubovoľnú gonion. funkciu na racionálnu.

príklad: $\int\frac{2-\sin x}{2+\cos x}dx = \left|\begin{matrix}\operatorname{tg}\frac{x}{2}=t\\dx=\frac{2}{1+t^2}dt\\\sin x=\frac{2t}{1+t^2}\\\cos x=\frac{1-t^2}{1+t^2}\end{matrix}\right| = \int\frac{2-\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt =$

$= \int\frac{\frac{2+2t^2-2t}{1+t^2}}{\frac{2+2t^2+1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt = \int\frac{2t^2-2t+2}{t^2+3}\cdot\frac{2}{1+t^2}dt = 4\int\frac{t^2-t+1}{(t^2+3)(1+t^2)}dt \overset{*}{=}$

$2(t^2-t+1)$; racion. funkcia :) rýdzoracion.

rozklad na parc. zlomky: $\frac{t^2-t+1}{(t^2+3)(t^2+1)} = \frac{At+B}{t^2+3}+\frac{Ct+D}{t^2+1}$ D.Ú.

⋮ $A=\frac12$, $B=1$, $D=0$, $C=-\frac12$

$* \ 4\int\left(\frac{\frac12 t+1}{t^2+3}+\frac{-\frac12 t}{t^2+1}\right)dt = \int\left(\frac{2t+4}{t^2+3}-\frac{2t}{t^2+1}\right)dt =$

$f(x) = t^2+3$
$f' = 2t$

$= \int\left(\frac{2t}{t^2+3}+\frac{4}{t^2+3}-\frac{2t}{t^2+1}\right)dt =$ (f', f; $3=a^2$ tabuľ. int. vzorec)

$\int\frac{1}{x^2+a^2}dx = \frac1a\operatorname{arctg}\frac{x}{a}+c$

$\operatorname{tg}\frac{x}{2} = t$

$= \ln(t^2+3)+\frac{4}{\sqrt3}\operatorname{arctg}\frac{t}{\sqrt3}-\ln(t^2+1)+c = \ln\left(\operatorname{tg}^2\frac{x}{2}+3\right)+\frac{4}{\sqrt3}\operatorname{arctg}\frac{\operatorname{tg}\frac{x}{2}}{\sqrt3}-\ln\left(\operatorname{tg}^2\frac{x}{2}+1\right)+c$