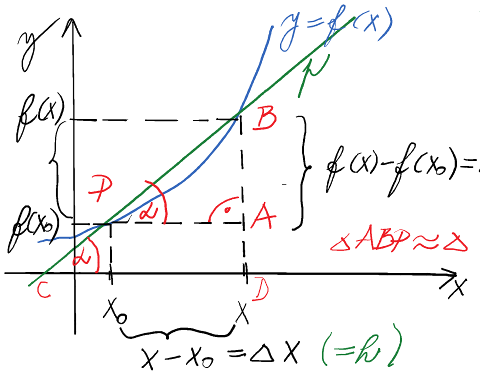
Derivácia, dotyčnica, diferenciál a L'Hospitalovo pravidlo

streda 5. novembra 2025 7:30

DOTYČNICA KU GRAFU FUNKCIE $y = f(x)$



t dotyčnica ku grafu funkcie v bode $P[x_0, f(x_0)]$ ✓

$f(x) - f(x_0) = \Delta f$ ✓

✓ $\triangle ABP$:

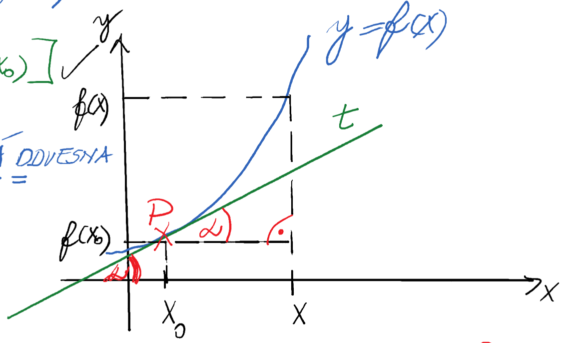
$\operatorname{tg}\alpha = \dfrac{\text{PROTIĽAHLÁ ODVESNA}}{\text{PRIĽAHLÁ}} = \dfrac{\Delta f}{\Delta x}$

$= k$

$\triangle ABP \approx \triangle DBC$

$x - x_0 = \Delta x \ (= h)$

$p: \ y = k \cdot x + q$ ✓

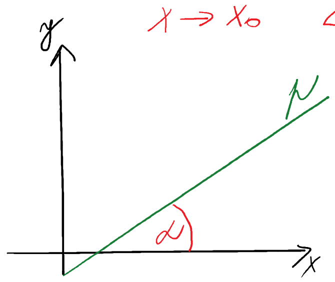


$x \to x_0 \qquad \Delta x \to 0$

$$f'(x_0) = \lim_{\substack{\Delta x \to 0 \\ h \to 0}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = k$$

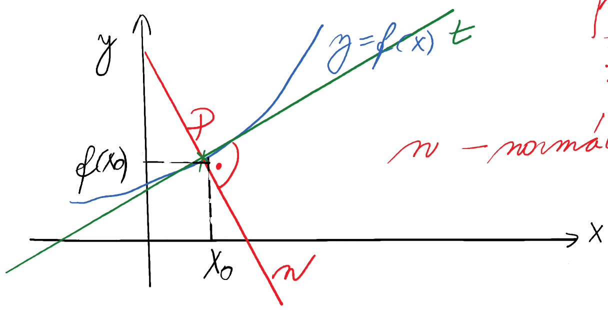
$$\boxed{k = f'(x_0)} \ ✓$$

$P = [x_0, f(x_0)]$



$y = kx + q$

$k = \operatorname{tg}\alpha$



n – normála

$t: \ y = k_t x + q_t$

$n: \ y = k_n x + q_n$

$t \perp n \Rightarrow k_t \cdot k_n = -1$

$k_n = -\dfrac{1}{k_t}$

$k_t = f'(x_0) \Rightarrow k_n = -\dfrac{1}{f'(x_0)}$

DOTYČNICA t: $\quad y - f(x_0) = f'(x_0)(x - x_0) \qquad P[x_0, f(x_0)]$

NORMÁLA n: $\quad y - f(x_0) = -\dfrac{1}{f'(x_0)} \cdot (x - x_0)$

PR $\quad f(x) = x \cdot \ln x \qquad D(f) = (0, \infty)$ ✓ $\quad \ln x^n = n \ln x$!

$P = [e^2, ?] \qquad x_0 = e^2$ ✓

t? $\ n$?

$f(x_0) = f(e^2) = e^2 \cdot \ln e^{2} = 2e^2 \boxed{\ln e} = 2e^2$ $\quad$ ($\ln e = 1$)

$P = [e^2, 2e^2]$ ✓ $\qquad \boxed{f(x_0) = 2e^2}$

($e^2 = x_0$)

$f'(x_0) = ?$ $\quad$ SÚČIN

$f'(x) = (x \cdot \ln x)' = 1 \ln x + x \cdot \dfrac{1}{x} = \ln x + 1$

$f'(x_0) = f'(e^2) = \ln e^{2} + 1 = 2 \ln e + 1 = 2 + 1 = 3$ $\quad$ ($\ln e = 1$)

$\boxed{f'(x_0) = 3} = k_t$

$y - f(x_0) = f'(x_0)(x - x_0)$

$t: \ y - 2e^2 = 3(x - e^2)$

$y = 3x \ldots e^2 \qquad y = kx + q$

$y - f(x_0) = f'(x_0)(x - x_0)$

t: $y - 2e^2 = 3(x - e^2)$

$y = \boxed{3}x - e^2$

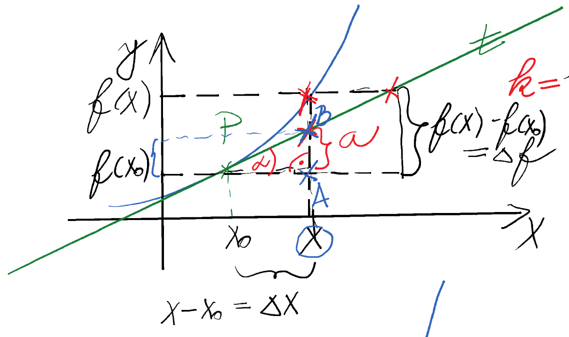
$y = kx + q$

$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)$

n: $y - 2e^2 = -\frac{1}{3}(x - e^2)$

$k_n = -\frac{1}{k_t}$

DIFERENCIÁL FUNKCIE



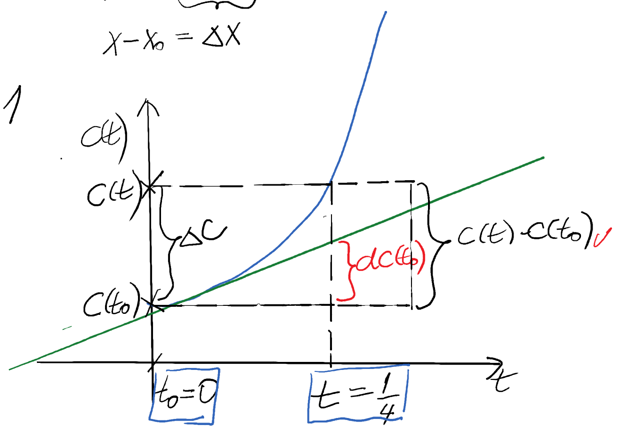
ΔABP

$k = tg\alpha = \frac{a}{\Delta x} = f'(x_0) \quad /\cdot \Delta x$

$a = f'(x_0)\Delta x = df(x_0)$

$a = df(x_0)$

PR 1



$\Delta x = x - x_0$

$\Delta t = t - t_0 = \frac{1}{4} - 0 = \frac{1}{4}$

$\Delta C = C(t) - C(t_0)$ PRESNE

$\Delta C \doteq dC(t_0) = C'(t_0)\Delta t$

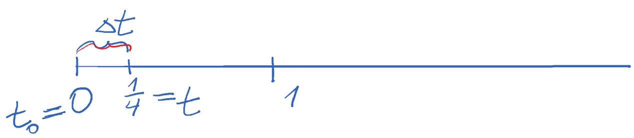
ODHAD

1) $t_0 = 0$

$t = \frac{3}{12} = \frac{1}{4}$

$dC(t_0) = ?$

$C(t) = 50t^2 + 100t + 10\,000$



$\Delta t = t - t_0$

$\Delta t = \frac{1}{4} - 0 = \frac{1}{4}$

$C'(t) = 100t + 100$

$C'(t_0) = C'(0) = 100$

$dC(t_0) = \underbrace{C'(t_0)}_{100} \cdot \underbrace{\Delta t}_{\frac{1}{4}} \quad (\doteq \Delta C)$

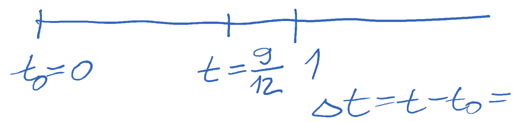
$dC(t_0) = 100 \cdot \frac{1}{4} = 25$ ks

POČAS NASLED. 3 MESIACOV VZRASTIE NÁKLAD O 25 KS. PRIBLIŽNE

2) $t_0 = 0$

$t = \frac{9}{12} = \frac{3}{4}$

$dC(t_0) = ?$



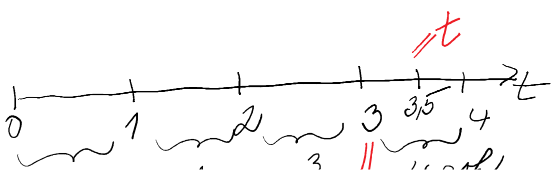
$\Delta t = t - t_0 = \frac{9}{12} - 0 = \frac{3}{4}$

$C'(t_0) = 100$

$dC(t_0) = \underbrace{100}_{C'(t_0)} \cdot \underbrace{\frac{3}{4}}_{\Delta t} = 75$ ks

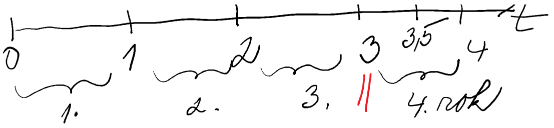
POČAS NASLEDUJÚCICH 9 MESIACOV VZRASTIE NÁKLAD O 75 KS. PRIBLIŽNE

3)



$\Delta t = t - t_0$

$\Delta t = 3{,}5 - 3 = 0{,}5$



$\Delta t = t - t_0$

$\Delta t = 3{,}5 - 3 = 0{,}5$

$t_0 = 3$
$t = 3{,}5$

$dC(t_0) = ?$

$dC(t_0) = C'(t_0)\Delta t$

$\Delta t = t - t_0 = 3{,}5 - 3 = \frac{1}{2}$

$C'(t) = 100t + 100$

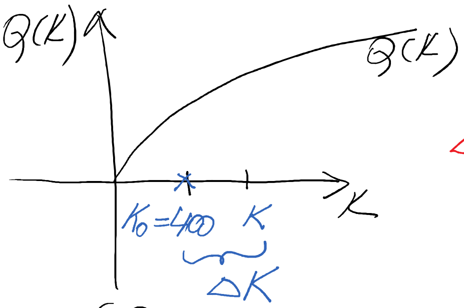
$C'(t_0) = C'(3) = 100 \cdot 3 + 100 = 400$

$dC(t_0) = C'(t_0)\cdot\Delta t = 400 \cdot \frac{1}{2} = 200$ ks. $\doteq \Delta C = C(t) - C(t_0)$

ODHAD

PRESNE: $\Delta C = C(t) - C(t_0) = C(3{,}5) - C(3) =$
$= 212{,}5 \doteq 212$ KS PRESNE

PR2: $Q(K) = 1200\sqrt{K} = 1200K^{\frac{1}{2}}$



1) $K_0 = 400$
$\Delta K = 10$

$dQ(K_0) = ?$

$\Delta K = K - K_0$
$K = K_0 + \Delta K$

$dQ(K_0) = Q'(K_0)\cdot\Delta K$

$Q'(K) = \cancel{1200}^{600} \cdot \frac{1}{2} \cdot K^{-\frac{1}{2}} = \frac{600}{\sqrt{K}}$

$Q'(K_0) = Q'(400) = \frac{600}{\sqrt{400}} = \frac{600}{20} = 30$

$dQ(K_0) = 30 \cdot 10 = 300$

DENNÁ PRODUKCIA VZRASTIE O 300 KS.

2) $\Delta K = -20$
$K_0 = 400$

$dQ(K_0) = ?$

$K = 380$

$\Delta K = K - K_0 = 380 - 400 = -20$ ✓

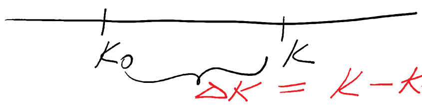
$Q'(K_0) = Q'(400) = 30$

$dQ(K_0) = 30 \cdot (-20) = -600$

DENNÁ PRODUKCIA SA ZNÍŽI O 600 KS.

3) $K_0 = 400$
$dQ(K_0) = 180$

$K = ?$ $(\Delta K = ?)$



$\Delta K = K - K_0$ ✓
$K = K_0 + \Delta K$ ✓

$dQ(K_0) = \underbrace{Q'(K_0)}_{30}\cdot\Delta K = 180$

$K = 6 + 400 = 406$

$30\,\Delta K = 180$

$\Delta K = 6$ (tisíc)

$K = 400 + 6 = 406$

KAP. INV. SA MUSÍ ZVÝŠIŤ NA HODNOTU 406 000

L'HOSPITALOVO PRAVIDLO

$X^3 - 4X^2 = X^2(X-4)$, $X^2 \to \infty$, $(X-4) \to \infty$

$\lim\limits_{x \to a} \frac{f(x)}{g(x)}$

$\lim\limits_{x \to 0} \frac{(\sin x)'}{x'}$

$\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} \exists$

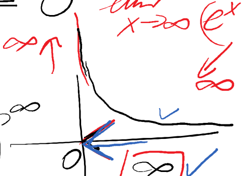
PR 1 $\lim\limits_{x \to 0} \frac{\sin x}{x} \overset{\boxed{\frac{0}{0}}}{\underset{L'H}{=}} \lim\limits_{x \to 0} \frac{\cos x}{1} = 1$, $\cos x \to 1$

PR2 a) $\lim\limits_{x \to \infty} \frac{x^3 - 4x^2}{x^2 - 1} \overset{\boxed{\frac{\infty}{\infty}}}{\underset{L'H}{=}} \lim\limits_{x \to \infty} \frac{3x^2 - 8x}{2x} \overset{\boxed{\frac{\infty}{\infty}}}{\underset{L'H}{=}} \lim\limits_{x \to \infty} \frac{6x - 8}{2} = \boxed{\infty}$, $6x - 8 \to \infty$, $2 \to 2$

b) $\lim\limits_{x \to \infty} \frac{5x^2 - 2x + 1}{3x^2 + 4x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim\limits_{x \to \infty} \frac{10x - 2}{6x + 4} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim \frac{10}{6} = \frac{5}{3}$

PR 3 $\lim\limits_{x \to \infty} \frac{x}{e^{x^2}} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim\limits_{x \to \infty} \frac{1}{e^{x^2} \cdot 2x}$ $\boxed{\frac{1}{\infty}}$ $= 0$

$\xrightarrow{L'H}$ $\lim\limits_{x \to \infty} \frac{0}{(e^{x^2} \cdot 2x)'}$, ∞, $= 0$



PR4 $\lim\limits_{x \to 0^+} x^2 e^{\frac{1}{x}}$, $x^2 \to 0$, $\frac{1}{x} \to \infty$, $e^{\frac{1}{x}} \to \infty$, $\boxed{0 \cdot \infty}$ $= \lim\limits_{x \to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x^2}}$, $e^{\frac{1}{x}} \to \infty$, $\frac{1}{x^2} \to \infty$, $= x^{-2}$, $\overset{\boxed{\frac{\infty}{\infty}}}{\underset{L'H}{=}}$, ∞, $\frac{1}{x^2}$

$= \lim\limits_{x \to 0^+} \frac{e^{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)'}{(x^{-2})'}$, $\frac{1}{x} = x^{-1}$, $\overset{L'H}{=} \lim\limits_{x \to 0^+} \frac{e^{\frac{1}{x}}(-1 \cdot x^{-2})}{-2x^{-3}} =$

$= \lim\limits_{x \to 0^+} \frac{e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}} = \lim\limits_{x \to 0^+} \frac{e^{\frac{1}{x}}}{2 \cdot \frac{1}{x}} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}$

$= \lim\limits_{x \to 0^+} \frac{e^{\frac{1}{x}} \cdot \cancel{\left(\frac{1}{x}\right)'}}{2 \cancel{\left(\frac{1}{x}\right)'}} = \lim\limits_{x \to 0^+} \frac{1}{2} \cdot e^{\frac{1}{x}} = \infty$, $e^{\frac{1}{x}} \to \infty$

PR5 $\lim\limits_{x \to \infty} \left(\frac{x^3 - 4x^2}{x^2 - 1} - x\right)$, $\frac{x^3 - 4x^2}{x^2 - 1} \to \infty$, $x \to \infty$, $\overset{\infty - \infty}{=}$

SPOLOČNÝ MENOVATEĽ

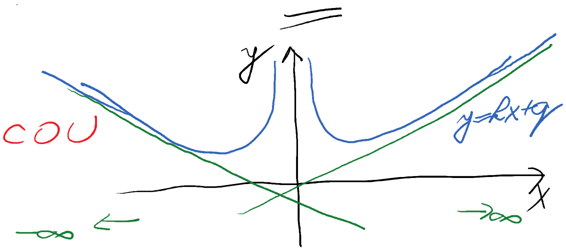
$\lim\limits_{x \to \infty} \frac{x^3 - 4x^2 - x(x^2 - 1)}{x^2 - 1} =$

$= \lim\limits_{x \to \infty} \frac{\cancel{x^3} - 4x^2 - \cancel{x^3} + x}{x^2 - 1} = \lim\limits_{x \to \infty} \frac{\boxed{-4}x^2 + x}{\boxed{1}x^2 - 1} \overset{\frac{-\infty}{\infty}}{\underset{L'H}{=}}$

$= \lim\limits_{x \to \infty} \frac{-8x + 1}{2x} \overset{\frac{-\infty}{\infty}}{\underset{L'H}{=}} \lim\limits_{x \to \infty} \frac{-8}{2} = -4$

y

ASYMPTOTY SO SMERNICOU



ASS: $y = kx + q$ ✓

$f(x) = \dfrac{x^3 - 4x^2}{x^2 - 1}$ ✓

$$k = \lim_{x \to \infty} \frac{f(x)}{x}$$

$$k = \lim_{x \to \pm\infty} \frac{\frac{x^3 - 4x^2}{x^2 - 1}}{\frac{x}{1}} = \lim_{x \to \pm\infty} \frac{x^3 - 4x^2}{x^3 - x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}$$

$$= \lim_{x \to \pm\infty} \frac{3x^2 - 8x}{3x^2 - 1} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x \to \pm\infty} \frac{6x - 8}{6x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}$$

$$= \lim_{x \to \pm\infty} \frac{6}{6} = 1 \in \mathbb{R} \ (\neq \pm\infty)$$

$$q = \lim_{x \to \infty} \left(f(x) - k \cdot x\right)$$

$$q = \lim_{x \to \pm\infty} \left(\frac{x^3 - 4x^2}{x^2 - 1} - 1 \cdot x\right) \overset{\infty - \infty}{=}$$

$$= \lim_{x \to \pm\infty} \frac{x^3 - 4x^2 - x(x^2 - 1)}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{\cancel{x^3} - 4x^2 - \cancel{x^3} + x}{x^2 - 1} =$$

$$= \ldots = -4 \ (\text{PRÍKLAD 5}) \in \mathbb{R} \ (\neq \pm\infty)$$

$y = 1 \cdot x - 4 \qquad x \to \pm\infty$

PR. $f(x) = x \cdot \ln x \qquad x > 0 \qquad D(f) = (0, \infty) = \mathbb{R}^+$

$\Rightarrow$ pre $x \to -\infty \ \nexists$ ASS

$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x \ln x}{x} = \infty \quad \nexists$ ASS $x \to \infty$

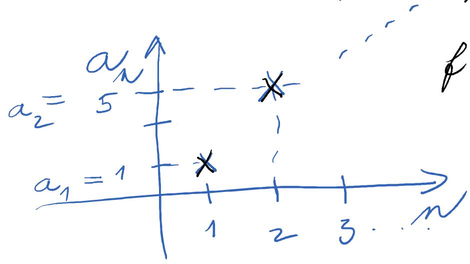
POSTUPNOSTI

1) $a_n = 3n - 1$ ✓

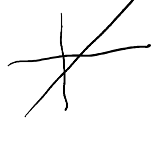
$a_1 = 3 \cdot 1 - 1 = 2$

$a_2 = 3 \cdot 2 - 1 = 5$

⋮



$f(x) = 3x - 1$



2) $a_n = 2^n$

GEOMETRICKÁ POSTUPNOSŤ

$a_1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$\dfrac{a_{n+1}}{a_n} = 2 = q$ QUOCIENT

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = a_1 \cdot \frac{1 - q^n}{1 - q} \quad \left(= a_1 \frac{q^n - 1}{q - 1}\right)$$

LIMITA POSTUPNOSTI

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \qquad 1^{\infty}$$

$$\lim_{n \to \infty} \left(1 + \frac{5}{n}\right)^n = \lim_{n \to \infty} \left(1 + \frac{1}{\frac{n}{5}}\right)^n =$$

$$\frac{n}{5} \to \infty = t \to \infty$$

$$n = 5t \; ✓$$

$$x^{a \cdot b} = (x^a)^b$$

$$= \lim_{t \to \infty} \left(1 + \frac{1}{t}\right)^{5t} = \left[\underbrace{\lim_{t \to \infty} \left(1 + \frac{1}{t}\right)^t}_{e}\right]^5 =$$

$$= [e]^5 = \underline{\underline{e^5}}$$

$$\lim_{n \to \infty} \left(1 + \frac{k}{n}\right)^n = e^k$$