FOURIEROV RAD (FR)

MR - TR

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

MÁME SYSTÉM $\{1, x, x^2, x^3, \dots\}$

$f(x)$ vyjadriť

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

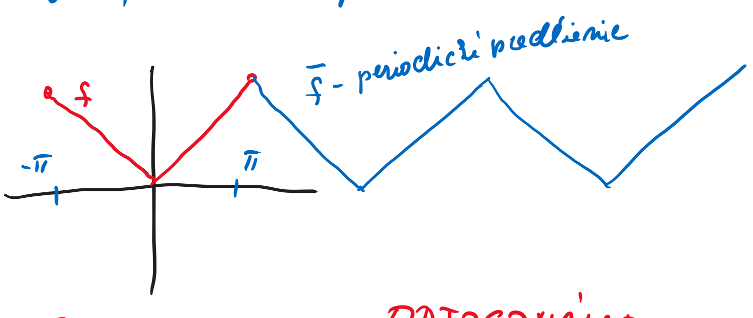
MÁME SYSTÉM GON. FCÍ

$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$

Chceme danú fciu $f(x)$ vyjadriť pomocou týchto fcií

$\longrightarrow f(x) = a_0 1 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$

$\cos nx, \sin nx$ sú periodické s periodou 2π t.j. aj $f(x)$ musí mať periódu 2π



Def. Funkcie $g(x), h(x)$ ... ORTOGONÁLNE na $\langle a,b \rangle$ ak platí

$$\int_a^b g(x) h(x)\, dx = 0$$

Náš systém $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$ je $\underbrace{\text{ORTOGONÁLNY}}_{\text{každá dvojica fcií}}$ na $\langle 0, 2\pi \rangle$

ZÁKLADNÁ VLASTNOSŤ, KTORÁ UMOŽNÍ NAJSŤ a_n, b_n

ad 1 pre 1. $\int_0^{2\pi} 1 \cdot \cos kx\, dx = \frac{\sin kx}{k} \Big|_0^{2\pi} = 0$

$k = 1, 2, 3, \dots$ $\int_0^{2\pi} 1 \cdot \sin kx\, dx = -\frac{\cos kx}{k} \Big|_0^{2\pi} = -(1-1) = 0$

ad 2 pre $\sin \alpha x, \cos \beta x$, $\alpha, \beta = 1, 2, 3, \dots$

$$\int_0^{2\pi} \sin \alpha x \cos \beta x\, dx = \frac{1}{2} \int_0^{2\pi} \sin \overbrace{(\alpha - \beta)}^{k} x + \sin \overbrace{(\alpha + \beta)}^{k} x\, dx = 0$$

$\sin \alpha x \ \sin \beta x$
$\alpha \neq \beta$

Veta (rozvoj do FR) Nech $f(x)$ je periodická (periódа 2π) (ohraničená). Potom

$$!! \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

kde $!! \ a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$ $\quad n = 0, 1, 2, 3, \dots$ $\qquad !! \ b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$ $\quad n = 1, 2, 3, \dots$

odvodenie pre a_1

$$!! \quad f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots \Big/ \cos x \int_0^{2\pi}$$

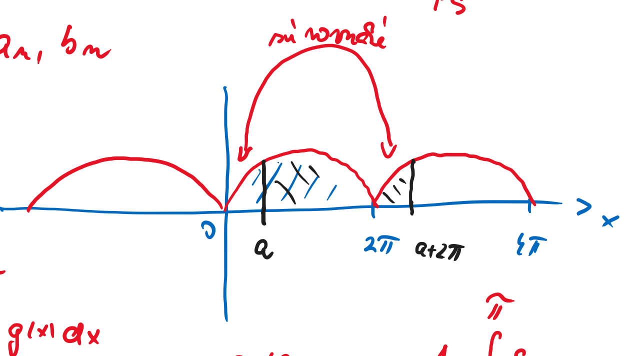
$$\underbrace{\int_0^{2\pi} f(x) \cos x}_{LS} = \frac{a_0}{2} \underbrace{\int_0^{2\pi} 1 \cos x}_{0} + a_1 \int_0^{2\pi} \cos^2 x\, dx + b_1 \underbrace{\int_0^{2\pi} \sin x \cos x}_{0} + a_2 \underbrace{\int_0^{2\pi} \cos x \cos 2x}_{0} + \dots$$

$$a_1 \int_0^{2\pi} \frac{1}{2} + \underbrace{\frac{\cos 2x}{2}}_{=0} dx = \frac{a_1}{2} x \Big|_0^{2\pi} = \underbrace{a_1 \pi}_{PS}$$

$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x\, dx$

TRIKY NA VÝPOČET a_n, b_n

ak $g(x)$ má periódu 2π

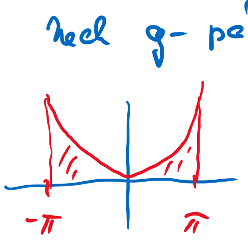


T1 $\underbrace{\int_0^{2\pi} g(x)\, dx}_{P} = \int_a^{a+2\pi} g(x)\, dx = \int_{-\pi}^{\pi} g(x)\, dx$ $\quad$ ak $a = -\pi$

mali by $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$ VÝHODNEJŠIE

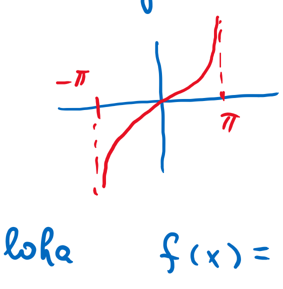
$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$

T2 Nech g - párna



$\int_{-\pi}^{\pi} g(x)\, dx = 2 \int_0^{\pi} g(x)\, dx$ $\leftarrow$ je lepšie dosadiť 0 ako $-\pi$ (malý pokrok)

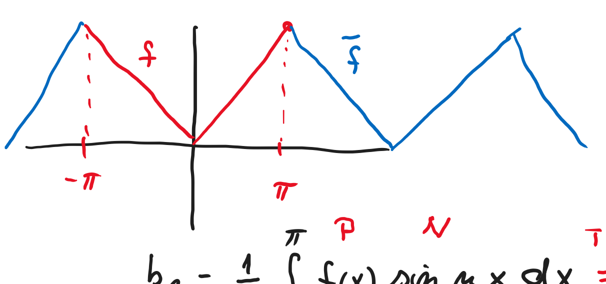
T3 Nech g - nepárna



$\int_{-\pi}^{\pi} g(x)\, dx = 0$ veľký pokrok

Úloha $f(x) = |x|$ na $(-\pi, \pi)$ Rozvinúť do FR

pre f musíme urobiť periodické pokr.



$$\bar{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + \underbrace{b_n}_{0} \sin nx \right)$$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \sin nx}_{N}\, dx \overset{T3}{=} 0$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x|\, dx \overset{T2}{=} \frac{2}{\pi} \int_0^{\pi} x\, dx = \frac{2}{\pi} \frac{x^2}{2} \Big|_0^{\pi} = \pi$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx\, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx\, dx = \frac{2}{\pi} \frac{(-1)^n - 1}{n^2}$

$$\int_0^{\pi} x \cos nx\, dx = \underbrace{x \frac{\sin nx}{n} \Big|_0^{\pi}}_{=0} - \int_0^{\pi} \frac{\sin nx}{n}\, dx = \frac{\cos nx}{n^2} \Big|_0^{\pi} = \frac{\cos n\pi - 1}{n^2} = \frac{(-1)^n - 1}{n^2}$$

n	$\cos n\pi$	$= (-1)^n$
1	-1	-1
2	1	1
3	-1	-1
⋮		

VIDÍM
f - párna $\Rightarrow$ FR = kosínusový
f - nepárna $\Rightarrow$ FR = sínusový

$$\tilde{f}(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$$

$\hookrightarrow$ koeficienty s párnym indexom sú 0

$\frac{(-1)^n - 1}{n^2} =$ párne $n = 2k$: 0; nepár. $n = 2k+1$: $\frac{-2}{(2k+1)^2}$

$$\tilde{f}(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos (2k+1) x$$

$x = 0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \boxed{\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots}$$

FR pre párnu period. fciu, nepárnu per. fciu

$! \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad ! \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

TEÓRIA

1. rady ČR, MR, TR, FR
2. DR
3. $\boxed{\text{FKP}}$

PRÍKLADY