Matematika 1 – 6.cvičenie

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Derivácia funkcie

Označenie derivácie funkcie v bode x_0: $f'(x_0)$

Pravidlá pre výpočet derivácie funkcie:

$$[cf(x)]' = cf'(x) \qquad c \in \mathbb{R}$$

$$[f(x) + g(x)]' = f'(x) + g'(x)$$

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

$$[f(g(x))]' = f'(g(x))g'(x).$$

$$\left[f(x)^{g(x)}\right]' = \left[e^{g(x)\cdot \ln f(x)}\right]'$$

Napríklad:

$(2x)'$

$(2x^2 + 5 - \arcsin x)'$

$(x.\ln x)'$

$\left(\frac{3x}{\sin x}\right)'$

$(\ln \sin 2x)'$

$(\sin x)^x = \left[e^{x.\ln(\sin x)}\right]'$

Derivácie elementárnych funkcií:

- $[c]' = 0 \quad [x^1]' = 1$
- $[x^\alpha]' = \alpha x^{\alpha-1}, \alpha \in R$
- $[\sin x]' = \cos x$
- $[\cos x]' = -\sin x$
- $[\operatorname{tg} x]' = \dfrac{1}{\cos^2 x}$
- $[\operatorname{cotg} x]' = -\dfrac{1}{\sin^2 x}$
- $[\arcsin x]' = \dfrac{1}{\sqrt{1-x^2}}$
- $[\arccos x]' = -\dfrac{1}{\sqrt{1-x^2}}$
- $[\operatorname{arctg} x]' = \dfrac{1}{1+x^2}$
- $[\operatorname{arccotg} x]' = -\dfrac{1}{1+x^2}$
- $[e^x]' = e^x$
- $[a^x]' = a^x \ln a$
- $[\ln x]' = \dfrac{1}{x}$
- $[\log_a x]' = \dfrac{1}{x \ln a}$

Ukážky použitia:

$[c]' = 0$ c – konštanta, bez x v zápise

$(2)' = 0$
$(-8)' = 0$

$(\ln 5)' = 0$
$(e^8)' = 0$
$(100^2)' = 0$
$\left(\frac{1}{50}\right)' = 0$

$[cf(x)]' = cf'(x)$ $c \in R$

$(2x)' = 2x'$

$(-8.\ln x)' = -8.(\ln x)'$
$(30.\sin x)' = 30.(\sin x)'$
$\left(\frac{1}{50}e^x\right)' = \frac{1}{50}(e^x)'$

$[x^\alpha]' = \alpha x^{\alpha-1}, \alpha \in R$

$[x^1]' = 1$

$(x^3)' = 3x^{3-1} = 3x^2$
$(2x)' = 2x' = 2.1 = 2$

Úprava výrazov na mocninový tvar

! 2^x *nie je typ* x^α !

$\left(\frac{1}{x^2}\right)' = (x^{-2})'$

$\left(\sqrt{x^3}\right)' = \left(x^{\frac{3}{2}}\right)'$

$\left(\frac{1}{\sqrt[5]{x}}\right)' = \left(x^{-\frac{1}{5}}\right)'$

Pr. 1: Vypočítajte deriváciu funkcie

$$f(x) = \sqrt[5]{x^3} + x^4 + \frac{1}{x^2} - 3\sqrt{x} + 8\frac{1}{\sqrt[4]{x}}$$

Pr. 2: Vypočítajte deriváciu funkcie

$$f(x) = 10x^3 + \frac{5}{x^4} - \frac{1}{\sqrt[9]{x}}$$

$$f'(x) = (10x^3)' + (5x^{-4})' - \left(x^{-\frac{1}{9}}\right)'$$

$$(c.f(x))' = c.\left(f(x)\right)'$$

$$f'(x) = 10(x^3)' + 5(x^{-4})' - \left(x^{-\frac{1}{9}}\right)'$$

$$(x^n)' = nx^{n-1}$$

$$f'(x) = 10.3.x^{3-1} + 5.(-4)x^{-4-1} - \left(\frac{-1}{9}x^{-\frac{1}{9}-1}\right)$$

$$f'(x) = 30x^2 - 20x^{-5} + \frac{1}{9}x^{-\frac{10}{9}}$$

$$f'(x) = 30.x^2 - \frac{20}{x^5} + \frac{1}{9\sqrt[9]{x^{10}}}$$

Pr. 3 – 25 /2: Vypočítajte deriváciu funkcie

$$f(x)=\sqrt[3]{x^4}+5^x-\ln x$$

$$f'(x) = (x^{\frac{4}{3}})' + (5^x)' - (\ln x)'$$

$$(x^n)' = nx^{n-1} \qquad (a^x)' = a^x.\ln a \qquad (\ln x)' = \frac{1}{x}$$

$$f'(x) = \frac{4}{3}x^{\frac{4}{3}-1} + 5^x.\ln 5 - \frac{1}{x}$$

$$f'(x) = \frac{4}{3}x^{\frac{1}{3}} + 5^x.\ln 5 - \frac{1}{x}$$

$$f'(x) = \frac{4}{3}\sqrt[3]{x} + 5^x.\ln 5 - \frac{1}{x}$$

Ukážky použitia:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

$$(\sin x \,.\, \ln x)' = (\sin x)'.\ln x + \sin x \,.\, (\ln x)'$$

$$(e^x.x^3)' = (e^x)'.x^3 + e^x.(x^3)'$$

$$[(x+1).\operatorname{tg} x]' = (x+1)'.\operatorname{tg} x + (x+1).(\operatorname{tg} x)'$$

Toto nederivujeme ako súčin funkcií:

$$(3(x+5))' = 3.(x+5)'$$

$$(2.\operatorname{cotg} x)' = 2(\operatorname{cotg} x)'$$

$$\left(\frac{\ln x}{2}\right)' = \frac{1}{2}.(\ln x)'$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$\left(\frac{\ln x}{x^2}\right)' = \frac{(\ln x)'.x^2 - \ln x \,.\, (x^2)'}{(x^2)^2}$$

$$\left(\frac{x}{\cos x}\right)' = \frac{(x)'.\cos x - x.(\cos x)'}{(x^2)^2}$$

$$\left(\frac{\operatorname{arctg} x}{\sqrt[5]{x}}\right)' = \frac{(\operatorname{arctg} x)'.\sqrt[5]{x} - \operatorname{arctg} x \,.\, \sqrt[5]{x})'}{(\sqrt[5]{x})^2}$$

Toto nederivujeme ako podiel funkcií:

$$\left(\frac{\ln x}{2}\right)' = \frac{1}{2}(\ln x)'$$

$$\left(\frac{3}{x^5}\right)' = 3\,(x^{-5})'$$

$$\left(\frac{1}{\sin x}\right)' = (\sin x^{-1})'$$

Pr. 4: Vypočítajte deriváciu funkcie

$$f(x) = \frac{\sin x}{2} + x.2^x + \frac{x}{\ln x} - \sqrt{x^3}$$

Pr. 5: Vypočítajte deriváciu funkcie

a) $f(x) = \dfrac{3^x - 8x + 3}{\operatorname{tg} x} + \log_3 x \,.\, e^x$

b) $f(x) = \dfrac{5^x}{x^3+1-\ln x} + \arcsin x \,.\, 8^x$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f(x)'.g(x) - f(x).g(x)'}{(g(x))^2} \qquad (f(x).g(x))' = f(x)'.g(x) + f(x).g(x)'$$

b)

$$f(x) = \frac{5^x}{x^3 + 1 - \ln x} + \arcsin x \,.\, 8^x$$

$$f'(x) = \frac{(5^x)'.(x^3+1-\ln x) - 5^x.(x^3+1-\ln x)'}{(x^3+1-\ln x)^2} + (\arcsin x)'.8^x + \arcsin x\,.(8^x)'$$

$$f'(x) = \frac{(5^x \ln 5).(x^3+1-\ln x) - 5^x.(3x^2+0-\frac{1}{x})}{(x^3+1-\ln x)^2} + \frac{1}{\sqrt{1-x^2}}.8^x + \arcsin x\,.(8^x \ln 8)$$

$$f'(x) = \frac{(5^x \ln 5).(x^3+1-\ln x) - 5^x.(3x^2-\frac{1}{x})}{(x^3+1-\ln x)^2} + \frac{1}{\sqrt{1-x^2}}.8^x + \arcsin x\,.(8^x \ln 8)$$

Ukážky použitia:

$$f\big(g(x)\big)' = f'(g(x)).\,g(x)'$$

Elementárna funkcia	Zložená funkcia
$\sin x$	$\sin(5x+6)$
$\ln x$	$\ln(\sqrt{3x})$
x^3	$(\cos x + 2x)^3$
$\arcsin x$	$\arcsin(x^3+1)$

V zloženej funkcii je x nahradené iným výrazom (funkciou).

Pr. 6: Vypočítajte deriváciu funkcie bez ďalších úprav

a) $f(x) = \ln(5x) + \sin(x+3)$

b) $f(x) = \operatorname{cotg}(5x^4 - 1) + \sin^3 x$

c) $f(x) = \ln\cos 8x + \sqrt{\sin(x-2)}$

d) $f(x) = \sqrt[3]{\frac{x+5}{x-4}} - 2.\log(x^2 - x - 2)$

e) $f(x) = \frac{\log_3 x}{x^2+5x} - 2.\arcsin(5x^4) + 2^x.\log_8 x$

f) $f(x) = \frac{\sqrt{x}-3^x}{\operatorname{arctg} x} + \cos(\ln 3x).(x^2 + 3x) - 10^{2025}$

e) $f(x) = \frac{\log_3 x}{x^2+5x} - 2.\arcsin(5x^4) + 2^x.\log_8 x$

$$\boldsymbol{f(g(x))' = f'(g(x)).g(x)'}$$

$$f'(x) = \frac{(\log_3 x)'.(x^2+5x) - \log_3 x\,.(x^2+5x)'}{(x^2+5x)^2} - 2(\arcsin'(5x^4)).(5x^4)' + (2^x)'.\log_8 x + 2^x(\log_8 x)'$$

$$f'(x) = \frac{\frac{1}{x\ln 3}.(x^2+5x) - \log_3 x\,.(2x+5)}{(x^2+5x)^2} - \frac{2}{\sqrt{1-(5x^4)^2}}.(5.4x^3) + (2^x \ln 2).\log_8 x + 2^x.\frac{1}{x\ln 8}$$

$$f'(x) = \frac{\frac{1}{x\ln 3}.(x^2+5x) - \log_3 x\,.(2x+5)}{(x^2+5x)^2} - \frac{40x^3}{\sqrt{1-(5x^4)^2}}.+(2^x \ln 2).\log_8 x + \frac{2^x}{x\ln 8}$$

f) $f(x) = \frac{\sqrt{x}-3^x}{\operatorname{arctg} x} + \cos(\ln 3x).(x^2+3x) - 10^{2025}$

$$f'(x) = \frac{(\sqrt{x}-3^x)'.(\operatorname{arctg} x) - (\sqrt{x}-3^x).(\operatorname{arctg} x)'}{(\operatorname{arctg} x)^2} +$$

$$+\cos'(\ln(3x)).(\ln 3x)'.(3x)'.(x^2+3x) + (\cos(\ln 3x)).(x^2+3x)' + (10^{2025})'$$

$$f'(x) = \frac{\left(\frac{1}{2}x^{-\frac{1}{2}} - 3^x \ln 3\right).(\operatorname{arctg} x) - (\sqrt{x}-3^x).\left(\frac{1}{1+x^2}\right)}{(\operatorname{arctg} x)^2} +$$

$$-\sin(\ln 3x).\frac{1}{3x}.3.(x^2+3x) + (\cos(\ln 3x)).(2x+3) + 0$$

$$f'(x) = \frac{\left(\frac{1}{2}\frac{1}{\sqrt{x}} - 3^x \ln 3\right).(\operatorname{arctg} x) - (\sqrt{x}-3^x).\left(\frac{1}{1+x^2}\right)}{(\operatorname{arctg} x)^2} +$$

$$-\sin(\ln 3x).\frac{1}{3x}.3.(x^2+3x) + (\cos(\ln 3x)).(2x+3) + 0$$

Pr. 7 – str. 26 / 27: Vypočítajte deriváciu funkcie

$$f(x) = x^{\sin x}$$

$$(f(x)^{g(x)})' = \left(e^{g(x).\ln f(x)}\right)'$$

$$f(x) = x^{\sin x}$$

$$g(x) = \sin x$$

$$f(x) = x$$

$$f(x) = e^{\sin x \,.\, \ln x}$$

$$f'(x) = (e^{\sin x \,.\, \ln x})'.\,(\sin x \,.\, \ln x)'$$

$$f'(x) = e^{\sin x \,.\, \ln x}.\,((\sin x)'.\,\ln x + \sin x \,.\,(\ln x)')$$

$$f'(x) = e^{\sin x \,.\, \ln x}.\,(\cos x \,.\, \ln x + \sin x \,.\frac{1}{x})$$

$$f'(x) = x^{\sin x}\,(\cos x \,.\, \ln x + \frac{\sin x}{x})$$

Dú : 25, 26 /1 - 35

Niektoré výsledky bez ďalších mat. úprav. Dú – derivácia funkcie

5. $f'(x) = 1 \cdot 10^{-x} + x \cdot 10^{-x} \ln 10 \cdot (-1)$
6. $f'(x) = \dfrac{\cos x \cdot (x-1) - \sin x \cdot 1}{(x-1)^2}$
7. $f'(x) = \dfrac{\frac{1}{\cos^2 x} \cdot x - \operatorname{tg} x \cdot 1}{x^2}$
8. $f'(x) = \dfrac{(2x+5)(x^2-5x+1) - (x^2+5x+2)(2x-5)}{(x^2-5x+1)^2}$
9. $f'(x) = \dfrac{1}{\sin 2x} \cdot \cos 2x \cdot 2$
10. $f'(x) = \dfrac{1}{\frac{5+4x}{\square}} \cdot \dfrac{4(3+7x) - (5+4x) \cdot 7}{(3+7x)^2}$

13. $f'(x) = \dfrac{1}{2} \cdot 2 \operatorname{tg} x \cdot \dfrac{1}{\cos^2 x} + 1 + \dfrac{1}{\cos^2 x}$

17. $f'(x) = -\dfrac{1}{\sqrt{1 - \left(\frac{3x-1}{4}\right)^2}} \cdot \dfrac{3}{4}$

18. $f'(x) = \dfrac{1}{1 + \left(x - \sqrt{1+x^2}\right)^2} \cdot \left(1 - \dfrac{1}{2\sqrt{1+x^2}} \cdot 2x\right)$

19. $f'(x) = \dfrac{1}{1 + \operatorname{tg}^2 x} \cdot \dfrac{1}{\cos^2 x}$

20. $f'(x) = \dfrac{1}{1 + \left(\frac{x+1}{x-1}\right)^2} \cdot \dfrac{1 \cdot (x-1) - (x+1) \cdot 1}{(x-1)^2}$

Príklady 25 – 31 riešte pomocou pomocného príkladu v tejto prezentácii (Pr. 7 na 16 snímke)

21. $f'(x) = 1 \cdot \arcsin x + x \cdot \dfrac{1}{\sqrt{1-x^2}} + \dfrac{1}{2\sqrt{1-x^2}} \cdot (-2x)$

22. $f'(x) = 1 \cdot \arccos x + x \cdot \left(-\dfrac{1}{\sqrt{1-x^2}}\right) - \dfrac{1}{2\sqrt{1-x^2}} \cdot (-2x)$

23. $f'(x) = 4\ln^3(\sin x) \cdot \dfrac{1}{\sin x} \cdot \cos x + \dfrac{1}{x^2-2x} \cdot (2x-2)$

24. $f'(x) = \dfrac{1}{3\left(\sqrt[3]{\ln \sin \frac{x}{2}}\right)^2} \cdot \dfrac{1}{\sin \frac{x}{2}} \cdot \cos \dfrac{x}{2} \cdot \dfrac{1}{2}$

32. $f'(x) = \dfrac{6x \cdot 3x^3 - (3x^2-1) \cdot 9x^2}{(3x^3)^2} + \dfrac{1}{\sqrt{1+x^2}} \cdot \dfrac{1}{2\sqrt{1+x^2}} \cdot 2x + \dfrac{1}{1+x^2}$

33. $f'(x) = \dfrac{1}{\cos\left(\operatorname{arctg} \frac{e^x-e^{-x}}{2}\right)} \cdot \left(-\sin\left(\operatorname{arctg} \dfrac{e^x-e^{-x}}{2}\right)\right) \cdot \dfrac{1}{1+\left(\frac{e^x-e^{-x}}{2}\right)^2} \cdot \dfrac{e^x-e^{-x} \cdot (-1)}{2}$

34. $f'(x) = 1 \cdot (\arcsin x)^2 + x \cdot 2\arcsin x \cdot \dfrac{1}{\sqrt{1-x^2}} - 2 +$
$2\left(\dfrac{1}{2\sqrt{1-x^2}} \cdot (-2x) \cdot \arcsin x + \sqrt{1-x^2} \cdot \dfrac{1}{\sqrt{1-x^2}}\right)$

35. $f'(x) = \dfrac{3}{4} \cdot \dfrac{1}{\frac{x^2+1}{x^2-1}} \cdot \dfrac{2x(x^2-1) - (x^2+1) \cdot 2x}{(x^2-1)^2} + \dfrac{1}{4} \cdot \dfrac{1}{\frac{x-1}{x+1}} \cdot \dfrac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} + \dfrac{1}{2} \cdot$
$\dfrac{1}{1+x^2}$