4 Estimation Theory

The task of estimation theory is to estimate as well as possible the unknown parameters of the distribution of the population from which a random sample comes. There are two types of estimates:

- A *point estimate* is an estimate given by a single value, obtained by replacing the unknown parameter with the value of a suitable sample characteristic.
- An *interval estimate* is an estimate given by an interval which, with a given probability, contains the value of the unknown parameter.

4.1 Point Estimate

A *point estimate* of a population parameter consists in replacing the unknown parameter with a single number, specifically a suitable sample characteristic obtained from the values of the random sample.

Let the population have a normal distribution with mean μ and standard deviation σ. Then the point estimate of the mean μ is the sample arithmetic mean $\overline{x}$, the point estimate of the variance σ^2 is the sample variance s^2, and the point estimate of the standard deviation σ is the sample standard deviation s. Similarly, if the population has an exponential distribution with mean λ, then the point estimate of the mean λ is the sample arithmetic mean $\overline{x}$.

Every good point estimate must satisfy the following properties:

- *unbiasedness* – under repeated sampling, the parameter estimate fluctuates symmetrically around the theoretical value on both sides,
- *consistency* – as the number of statistical units in the random sample increases, the estimate becomes increasingly concentrated around the estimated value,
- *efficiency* – under repeated sampling, the variance of the estimate is minimal among all possible estimates,
- *robustness* – outlying values, caused for example by gross measurement error, do not have a large effect on the value of the estimate.

4.2 Interval Estimate

An *interval estimate* of a population parameter consists in finding a confidence interval which, with a given probability γ, contains the value of the unknown parameter. This probability

γ is called the *confidence coefficient* of the estimate. The number $\alpha = 1 - \gamma$ is called the *significance level* and indicates the error rate we are willing to accept in the interval estimate of the parameter, i.e., in what percentage of cases the true value of the parameter will not lie within the interval. The significance levels most often chosen are 0.01 and 0.05. The interval obtained is then called the $100(1-\alpha)\%$ *confidence interval*, and in the most common cases it is the 99%, resp. 95% confidence interval.

A confidence interval can take three forms. It can be two-sided (bounded on both the right and the left) or one-sided (bounded only on one side, either left or right).
A *two-sided confidence interval* for a parameter Q of the population is an estimate given by the numerical interval $\langle Q_d; Q_h \rangle$, for which the probability satisfies

$$P(Q_d \leqq Q \leqq Q_h) = 1 - \alpha, \text{ where } \alpha \in (0; 1).$$

A *left-sided confidence interval* for a parameter Q of the population is an estimate given by the numerical interval $\langle Q_d; \infty)$, for which the probability satisfies

$$P(Q_d \leqq Q) = 1 - \alpha, \text{ where } \alpha \in (0; 1).$$

A *right-sided confidence interval* for a parameter Q of the population is an estimate given by the numerical interval $(-\infty; Q_h \rangle$, for which the probability satisfies

$$P(Q \leqq Q_h) = 1 - \alpha, \text{ where } \alpha \in (0; 1).$$

4.2.1 Confidence Interval for Estimating Parameters of the Normal Distribution

Consider a random sample V_n coming from a population with a normal distribution with mean μ and standard deviation σ. We write $V_n \sim N(\mu, \sigma^2)$.

When determining the confidence interval for individual parameters, we will distinguish the following cases:

- estimation of the mean μ, if we know the variance σ^2,
- estimation of the mean μ, if we do not know the variance σ^2,
- estimation of the variance σ^2, if we know the mean μ,
- estimation of the variance σ^2, if we do not know the mean μ.

When constructing the confidence interval we will make use of the concept of a *suitable statistic*. A *statistic* $g(x)$ is a function of the random sample whose distribution does not depend on the parameters of the distribution from which the random sample comes.

The procedure for constructing a confidence interval for a parameter Q is as follows:

1. Determine the point estimate of the parameter Q.
2. Complete this estimate into a suitable statistic $g(x)$.
3. Determine numbers g_1, g_2, $g_1 < g_2$ such that $P(g \leqq g_1) = \alpha_1$, $P(g_2 \leqq g) = \alpha_2$, where $\alpha_1 + \alpha_2 = \alpha$, $\alpha_1, \alpha_2 \in (0; 1)$. If the distribution function $F(x)$ is continuous and increasing, then g_1, g_2 are quantiles of the corresponding distribution, i.e. $g_1 = F^{-1}(\alpha_1)$, $g_2 = F^{-1}(1 - \alpha_2)$.
4. By adding the equations $P(g \leqq g_1) = \alpha_1$, $P(g_2 \leqq g) = \alpha_2$ and applying subsequent equivalent transformations, we obtain the appropriate confidence interval.

Confidence interval for estimating the mean μ, if we know the variance σ^2

In the case of a two-sided confidence interval we proceed as follows:

1. The point estimate of the mean μ is the sample arithmetic mean (71), resp. (72).
2. A suitable statistic is
$$g(x) = \frac{\overline{x} - \mu}{\sigma} \sqrt{n} \sim N(0, 1).$$
3. The numbers $g_1 = u_{\frac{\alpha}{2}}$, $g_2 = u_{1-\frac{\alpha}{2}}$ are quantiles of the standard normal distribution (Table 28).
4. Since for the two-sided confidence interval we require $P(g_1 \leqq g \leqq g_2) = 1 - \alpha$, we obtain
$$u_{\frac{\alpha}{2}} \leqq \frac{\overline{x} - \mu}{\sigma} \sqrt{n} \leqq u_{1-\frac{\alpha}{2}}.$$
By equivalent transformations, together with the fact that the normal distribution is symmetric, i.e. $u_{\frac{\alpha}{2}} = -u_{1-\frac{\alpha}{2}}$ holds, we obtain the following $100(1 - \alpha)\%$ two-sided confidence interval for estimating the mean μ, if we know the variance σ^2
$$\mu \in \left\langle \overline{x} - u_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}; \overline{x} + u_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right\rangle. \tag{78}$$

We proceed similarly when determining the left-sided and right-sided confidence intervals with probability $1 - \alpha$. Thus, the $100(1 - \alpha)\%$ left-sided confidence interval for the mean μ, if we know the variance σ^2, has the form
$$\mu \in \left\langle \overline{x} - u_{1-\alpha} \frac{\sigma}{\sqrt{n}}; \infty \right) \tag{79}$$

and the $100(1-\alpha)\%$ right-sided confidence interval for the mean μ, if we know the variance σ^2, has the form

$$\mu \in \left\langle -\infty; \overline{x} + u_{1-\alpha} \frac{\sigma}{\sqrt{n}} \right\rangle. \tag{80}$$

Example 4.1 *Consider a random variable with variance σ^2=7.4, coming from a population with a normal distribution. The values obtained from the random sample are given in the following table:*

x_j	9	11	12	14	15	16	17	18	20	21
n_j	1	2	3	4	7	5	4	3	2	1

Determine

a) the 99% two-sided confidence interval for the unknown mean,

b) the 90% left-sided confidence interval for the unknown mean,

c) the 95% right-sided confidence interval for the unknown mean.

Solution: Based on the problem statement, it is clear that we are to determine confidence intervals for "estimating the mean when the variance is known". The sample size is $n = 32$. First we need a suitable point estimate of the mean μ, which is the sample arithmetic mean $\overline{x}$ calculated according to relation (72)

$$\overline{x} = \frac{1}{32}(9 \cdot 1 + 11 \cdot 2 + 12 \cdot 3 + 14 \cdot 4 + 15 \cdot 7 + 16 \cdot 5 + 17 \cdot 4 + 18 \cdot 3 + 20 \cdot 2 + 21 \cdot 1) = 15,344.$$

a) To determine the two-sided confidence interval we use relation (78). In it we need to determine $\alpha = 0,01$, $\sigma = \sqrt{7,4} = 2,72$ and the normal distribution quantile $u_{1-\frac{\alpha}{2}} = u_{0,995} = 2,576$ (Table 28). Substituting into relation (78) we obtain the 99% two-sided confidence interval for the unknown mean μ

$$\mu \in \left\langle 15,544 - 2,576 \cdot \frac{2,72}{\sqrt{32}}; 15,544 + 2,576 \cdot \frac{2,72}{\sqrt{32}} \right\rangle \Rightarrow \mu \in \langle 14,11; 16,58 \rangle .$$

b) To determine the left-sided confidence interval we use relation (79). In it we need to determine $\alpha = 0,1$, $\sigma = \sqrt{7,4} = 2,72$ and the normal distribution quantile $u_{1-\alpha} = u_{0,9} = 1,282$ (Table 28). Substituting into relation (79) we obtain the 90% left-sided confidence interval for the unknown mean μ

$$\mu \in \left\langle 15,544 - 1,282 \cdot \frac{2,72}{\sqrt{32}}; \infty \right\rangle \Rightarrow \mu \in \langle 14,727; \infty \rangle .$$

c) To determine the right-sided confidence interval we use relation (80). In it we need to determine $\alpha = 0,05$, $\sigma = \sqrt{7,4} = 2,72$ and the normal distribution quantile $u_{1-\alpha} = u_{0,95} =$

$1,645$ (Table 28). Substituting into relation (80) we obtain the 95% right-sided confidence interval for the unknown mean μ

$$\mu \in \left(-\infty; 15,544 + 1,645 \cdot \frac{2,72}{\sqrt{32}} \right\rangle \Rightarrow \mu \in (-\infty; 16,135\rangle \,.$$

Confidence interval for estimating the mean μ, if we do not know the variance σ^2

In the case of a two-sided confidence interval, if the random sample size is small ($n < 30$), we proceed as follows:

1. The point estimate of the mean μ is the sample arithmetic mean (71), resp. (72).

2. A suitable statistic is
$$g(x) = \frac{\overline{x} - \mu}{s} \sqrt{n} \sim t(n-1).$$

3. The numbers $g_1 = t_{\frac{\alpha}{2}}(n-1)$, $g_2 = t_{1-\frac{\alpha}{2}}(n-1)$ are quantiles of Student's t-distribution (Table 29).

4. Since for the two-sided confidence interval we require $P(g_1 \leqq g \leqq g_2) = 1-\alpha$, we obtain
$$t_{\frac{\alpha}{2}}(n-1) \leqq \frac{\overline{x} - \mu}{s} \cdot \sqrt{n} \leqq t_{1-\frac{\alpha}{2}}(n-1).$$
By equivalent transformations, together with the fact that Student's t-distribution is symmetric, i.e. $t_{\frac{\alpha}{2}}(n-1) = -t_{1-\frac{\alpha}{2}}(n-1)$ holds, we obtain the following $100(1-\alpha)\%$ two-sided confidence interval for estimating the mean μ, if we do not know the variance σ^2
$$\mu \in \left\langle \overline{x} - t_{1-\frac{\alpha}{2}}(n-1) \frac{s}{\sqrt{n}}; \overline{x} + t_{1-\frac{\alpha}{2}}(n-1) \frac{s}{\sqrt{n}} \right\rangle . \tag{81}$$

We proceed similarly when determining the left-sided and right-sided confidence intervals with probability $1-\alpha$. Thus, the $100(1-\alpha)\%$ left-sided confidence interval for the mean μ, if we do not know the variance σ^2, has the form

$$\mu \in \left\langle \overline{x} - t_{1-\alpha}(n-1) \frac{s}{\sqrt{n}}; \infty \right) \tag{82}$$

and the $100(1-\alpha)\%$ right-sided confidence interval for the mean μ, if we do not know the variance σ^2, has the form

$$\mu \in \left(-\infty; \overline{x} + t_{1-\alpha}(n-1) \frac{s}{\sqrt{n}} \right\rangle . \tag{83}$$

If the random sample size is sufficiently large ($n \geqq 30$), we proceed in the same way, whereby the statistic $g(x) = \frac{\overline{x}-\mu}{s}\sqrt{n}$ will have the normal distribution $N(0,1)$, as in the first case. Then the $100(1-\alpha)\%$ two-sided confidence interval for estimating the mean μ, if we do not know the variance σ^2, has the form

$$\mu \in \left\langle \overline{x} - u_{1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}}; \overline{x} + u_{1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \right\rangle. \tag{84}$$

The $100(1-\alpha)\%$ left-sided confidence interval for the mean μ, if we do not know the variance σ^2, has the form

$$\mu \in \left\langle \overline{x} - u_{1-\alpha} \frac{s}{\sqrt{n}}; \infty \right) \tag{85}$$

and the $100(1-\alpha)\%$ right-sided confidence interval for the mean μ, if we do not know the variance σ^2, has the form

$$\mu \in \left(-\infty; \overline{x} + u_{1-\alpha} \frac{s}{\sqrt{n}} \right\rangle. \tag{86}$$

Example 4.2 *Consider a random variable coming from a population with a normal distribution. The values obtained from the random sample are given in the following table:*

x_j	9	11	12	14	15	16	17	18	20	21
n_j	1	2	3	3	6	4	4	3	2	1

Determine

a) the 90% *two-sided confidence interval for the unknown mean,*

b) the 99% *left-sided confidence interval for the unknown mean,*

c) the 95% *right-sided confidence interval for the unknown mean.*

Solution: Based on the problem statement, it is clear that we are to determine confidence intervals for "estimating the mean when the variance is unknown". The sample size is $n = 29$. First we need a suitable point estimate of the mean μ, which is the sample arithmetic mean $\overline{x}$ calculated according to relation (72)

$$\overline{x} = \frac{1}{29}(9 \cdot 1 + 11 \cdot 2 + 12 \cdot 3 + 14 \cdot 3 + 15 \cdot 6 + 16 \cdot 4 + 17 \cdot 4 + 18 \cdot 3 + 20 \cdot 2 + 21 \cdot 1) = 15,379.$$

Since we do not know the variance σ^2, we will need the sample variance s^2 and the sample standard deviation s, which we calculate according to relations (74) and (77)

$$s^2 = \frac{1}{28}\big[1 \cdot (9 - 15,379)^2 + 2 \cdot (11 - 15,379)^2 + 3 \cdot (12 - 15,379)^2 + 3 \cdot (14 - 15,379)^2 +$$

$$+6 \cdot (15 - 15,379)^2 + 4 \cdot (16 - 15,379)^2 + 4 \cdot (17 - 15,379)^2 + 3 \cdot (18 - 15,379)^2 +$$

$$+2 \cdot (20 - 15,379)^2 + 1 \cdot (21 - 15,379)^2\big] = 8,101 \Rightarrow s = \sqrt{s^2} = 2,846.$$

Since the sample size is $n < 30$, we use relations (81), (82) and (83) to determine the confidence intervals.

a) For the two-sided confidence interval we need to determine $\alpha = 0,1$ and the Student's t-distribution quantile $t_{1-\frac{\alpha}{2}}(n-1) = t_{0,95}(28) = 1,701$ (Table 29). Substituting into relation (81) we obtain the 90% two-sided confidence interval for the unknown mean μ

$$\mu \in \left\langle 15,379 - 1,701 \cdot \frac{2,846}{\sqrt{29}}; 15,379 + 1,701 \cdot \frac{2,846}{\sqrt{29}} \right\rangle \Rightarrow \mu \in \langle 14,480; 16,278 \rangle .$$

b) For the left-sided confidence interval we need to determine $\alpha = 0,01$ and the Student's t-distribution quantile $t_{1-\alpha}(n-1) = t_{0,99}(28) = 2,467$ (Table 29). Substituting into relation (82) we obtain the 99% left-sided confidence interval for the unknown mean μ

$$\mu \in \left\langle 15,379 - 2,467 \cdot \frac{2,846}{\sqrt{29}}; \infty \right) \Rightarrow \mu \in \langle 14,075; \infty) .$$

c) For the right-sided confidence interval we need to determine $\alpha = 0,05$ and the Student's t-distribution quantile $t_{1-\alpha}(n-1) = t_{0,95}(28) = 1,701$ (Table 29). Substituting into relation (83) we obtain the 95% right-sided confidence interval for the unknown mean μ

$$\mu \in \left(-\infty; 15,379 + 1,701 \cdot \frac{2,846}{\sqrt{29}} \right\rangle \Rightarrow \mu \in (-\infty; 16,278 \rangle .$$

Confidence interval for estimating the variance σ^2, if we know the mean μ

In the case of a two-sided confidence interval we proceed as follows:

1. The point estimate of the variance σ^2 is the sample variance s_0^2 (75), resp. (76).

2. A suitable statistic is
$$g(x) = \frac{n\, s_0^2}{\sigma^2} \sim \chi^2(n).$$

3. The numbers $g_1 = \chi^2_{\frac{\alpha}{2}}(n)$, $g_2 = \chi^2_{1-\frac{\alpha}{2}}(n)$ are quantiles of the χ^2-distribution (Tables 30 – 32).

4. Since for the two-sided confidence interval we require $P(g_1 \leqq g \leqq g_2) = 1-\alpha$, we obtain
$$\chi^2_{\frac{\alpha}{2}}(n) \leqq \frac{n\, s_0^2}{\sigma^2} \leqq \chi^2_{1-\frac{\alpha}{2}}(n).$$

By equivalent transformations we obtain the following $100(1-\alpha)\%$ two-sided confidence interval for estimating the variance σ^2, if we know the mean μ

$$\sigma^2 \in \left\langle \frac{n\, s_0^2}{\chi^2_{1-\frac{\alpha}{2}}(n)}; \frac{n\, s_0^2}{\chi^2_{\frac{\alpha}{2}}(n)} \right\rangle. \tag{87}$$

We proceed similarly when determining the left-sided and right-sided confidence intervals with probability $1-\alpha$. Thus, the $100(1-\alpha)\%$ left-sided confidence interval for the variance σ^2, if we know the mean μ, has the form

$$\sigma^2 \in \left\langle \frac{n\, s_0^2}{\chi^2_{1-\alpha}(n)}; \infty \right) \tag{88}$$

and the $100(1-\alpha)\%$ right-sided confidence interval for the variance σ^2, if we know the mean μ, has the form

$$\sigma^2 \in \left(0; \frac{n\, s_0^2}{\chi^2_{\alpha}(n)} \right\rangle. \tag{89}$$

Example 4.3 *Consider a random variable coming from a population with a normal distribution and mean μ=15. The values obtained from the random sample are given in the following table:*

x_j	9	11	12	14	15	16	17	18	20	21
n_j	1	2	3	4	7	5	4	3	2	1

Determine

a) the 99% *two-sided confidence interval for the variance,*

b) the 95% *left-sided confidence interval for the variance,*

c) the 95% *right-sided confidence interval for the variance.*

Solution: Based on the problem statement, it is clear that we are to determine confidence intervals for "estimating the variance when the mean is known". The sample size is $n = 32$. First we need a suitable point estimate of the variance σ^2, which is the sample variance s_0^2. Since we know the mean μ, we use relation (76)

$$s_0^2 = \frac{1}{32}\big[1\cdot(9-15)^2+2\cdot(11-15)^2+3\cdot(12-15)^2+4\cdot(14-15)^2+7\cdot(15-15)^2+5\cdot(16-15)^2+$$

$$+4\cdot(17-15)^2+3\cdot(18-15)^2+2\cdot(20-15)^2+1\cdot(21-15)^2\big] = 7,281.$$

a) To determine the two-sided confidence interval we use relation (87). In it we need to determine $\alpha = 0,01$ and the χ^2-distribution quantiles $\chi^2_{\frac{\alpha}{2}}(n) = \chi^2_{0,005}(32) = 15,13$; $\chi^2_{1-\frac{\alpha}{2}}(n) =$

$\chi^2_{0,995}(32) = 56,33$ (Table 30). Substituting into relation (87) we obtain the 99% two-sided confidence interval for the variance σ^2

$$\sigma^2 \in \left\langle \frac{32 \cdot 7,281}{56,33}; \frac{32 \cdot 7,281}{15,13} \right\rangle \Rightarrow \sigma^2 \in \langle 4,136; 15,396 \rangle .$$

b) To determine the left-sided confidence interval we use relation (88). In it we need to determine $\alpha = 0,05$ and the χ^2-distribution quantile $\chi^2_{1-\alpha}(n) = \chi^2_{0,95}(32) = 46,19$ (Table 30). Substituting into relation (88) we obtain the 95% left-sided confidence interval for the variance σ^2

$$\sigma^2 \in \left\langle \frac{32 \cdot 7,281}{46,19}; \infty \right) \Rightarrow \sigma^2 \in \langle 5,044; \infty) .$$

c) To determine the right-sided confidence interval we use relation (89). In it we need to determine $\alpha = 0,05$ and the χ^2-distribution quantile $\chi^2_{\alpha}(n) = \chi^2_{0,05}(32) = 20,07$ (Table 30). Substituting into relation (89) we obtain the 95% right-sided confidence interval for the variance σ^2

$$\sigma^2 \in \left(0; \frac{32 \cdot 7,281}{20,07} \right\rangle \Rightarrow \sigma^2 \in (0; 11,608 \rangle .$$

Confidence interval for estimating the variance σ^2, if we do not know the mean μ

In the case of a two-sided confidence interval we proceed as follows:

1. The point estimate of the variance σ^2 is the sample variance s^2 (73), resp. (74).

2. A suitable statistic is
$$g(x) = \frac{(n-1)\, s^2}{\sigma^2} \sim \chi^2(n-1).$$

3. The numbers $g_1 = \chi^2_{\frac{\alpha}{2}}(n-1)$, $g_2 = \chi^2_{1-\frac{\alpha}{2}}(n-1)$ are quantiles of the χ^2-distribution (Tables 30 – 32).

4. Since for the two-sided confidence interval we require $P(g_1 \leqq g \leqq g_2) = 1-\alpha$, we obtain
$$\chi^2_{\frac{\alpha}{2}}(n-1) \leqq \frac{(n-1)\, s^2}{\sigma^2} \leqq \chi^2_{1-\frac{\alpha}{2}}(n-1).$$
By equivalent transformations we obtain the following $100(1-\alpha)\%$ two-sided confidence interval for estimating the variance σ^2, if we do not know the mean μ
$$\sigma^2 \in \left\langle \frac{(n-1)\, s^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}; \frac{(n-1)\, s^2}{\chi^2_{\frac{\alpha}{2}}(n-1)} \right\rangle . \tag{90}$$

We proceed similarly when determining the left-sided and right-sided confidence intervals with probability $1-\alpha$. Thus, the $100(1-\alpha)\%$ left-sided confidence interval for the variance σ^2, if we do not know the mean μ, has the form

$$\sigma^2 \in \left\langle \frac{(n-1)\,s^2}{\chi^2_{1-\alpha}(n-1)}; \infty \right) \tag{91}$$

and the $100(1-\alpha)\%$ right-sided confidence interval for the variance σ^2, if we do not know the mean μ, has the form

$$\sigma^2 \in \left(0; \frac{(n-1)\,s^2}{\chi^2_{\alpha}(n-1)} \right\rangle . \tag{92}$$

Example 4.4 *Consider a random variable coming from a population with a normal distribution. The values obtained from the random sample are given in the following table:*

x_j	9	11	12	14	15	16	17	18	20	21
n_j	1	2	3	4	7	5	4	3	2	1

Determine

a) the 99% *two-sided confidence interval for the variance,*

b) the 95% *left-sided confidence interval for the variance,*

c) the 95% *right-sided confidence interval for the variance.*

Solution: Based on the problem statement, it is clear that we are to determine confidence intervals for "estimating the variance when the mean is unknown". The sample size is $n = 32$. First we need a suitable point estimate of the variance σ^2, which is the sample variance s^2. Since we do not know the mean μ, we use relation (74), where the sample arithmetic mean $\overline{x}$ is calculated according to relation (72)

$$\overline{x} = \frac{1}{32}(9 \cdot 1 + 11 \cdot 2 + 12 \cdot 3 + 14 \cdot 4 + 15.7 + 16 \cdot 5 + 17 \cdot 4 + 18 \cdot 3 + 20 \cdot 2 + 21 \cdot 1) = 15,344.$$

Then for the sample variance s^2, based on relation (74), we obtain

$$s^2 = \frac{1}{31}\big[1 \cdot (9-15,344)^2 + 2 \cdot (11-15,344)^2 + 3 \cdot (12-15,344)^2 + 4 \cdot (14-15,344)^2 +$$

$$+7 \cdot (15-15,344)^2 + 5 \cdot (16-15,344)^2 + 4 \cdot (17-15,344)^2 + 3 \cdot (18-15,344)^2 +$$

$$+2 \cdot (20-15,344)^2 + 1 \cdot (21-15,344)^2\big] = 7,394.$$

a) To determine the two-sided confidence interval we use relation (90). In it we need to determine $\alpha = 0,01$ and the χ^2-distribution quantiles $\chi^2_{\frac{\alpha}{2}}(n-1) = \chi^2_{0,005}(31) = 14,46$;

$\chi^2_{1-\frac{\alpha}{2}}(n-1) = \chi^2_{0,995}(31) = 55,00$ (Table 30). Substituting into relation (90) we obtain the 99% two-sided confidence interval for the variance σ^2

$$\sigma^2 \in \left\langle \frac{31 \cdot 7,394}{55,00}; \frac{31 \cdot 7,394}{14,46} \right\rangle \Rightarrow \sigma^2 \in \langle 4,1674; 15,8544 \rangle .$$

b) To determine the left-sided confidence interval we use relation (91). In it we need to determine $\alpha = 0,05$ and the χ^2-distribution quantile $\chi^2_{1-\alpha}(n-1) = \chi^2_{0,95}(31) = 44,99$ (Table 30). Substituting into relation (91) we obtain the 95% left-sided confidence interval for the variance σ^2

$$\sigma^2 \in \left\langle \frac{31 \cdot 7,394}{44,99}; \infty \right) \Rightarrow \sigma^2 \in \langle 5,0954; \infty) .$$

c) To determine the right-sided confidence interval we use relation (92). In it we need to determine $\alpha = 0,05$ and the χ^2-distribution quantile $\chi^2_{\alpha}(n-1) = \chi^2_{0,05}(31) = 19,28$ (Table 30). Substituting into relation (92) we obtain the 95% right-sided confidence interval for the variance σ^2

$$\sigma^2 \in \left(0; \frac{31 \cdot 7,394}{19,28} \right\rangle \Rightarrow \sigma^2 \in (0; 11,8886\rangle .$$

4.2.2 Confidence Interval for Estimating the Parameter of the Exponential Distribution

Consider a random sample V_n coming from a population with an exponential distribution with parameter λ, $\lambda > 0$. We write $V_n \sim \mathrm{Ex}(\lambda)$. In the case of a two-sided confidence interval we proceed as follows:

1. The point estimate of the parameter λ is the sample arithmetic mean (71), resp. (72).

2. A suitable statistic is
$$g(x) = \frac{2n\overline{x}}{\lambda} \sim \chi^2(2n).$$

3. The numbers $g_1 = \chi^2_{\frac{\alpha}{2}}(2n)$, $g_2 = \chi^2_{1-\frac{\alpha}{2}}(2n)$ are quantiles of the χ^2-distribution (Tables 30 – 32).

4. Since for the two-sided confidence interval we require $P(g_1 \leqq g \leqq g_2) = 1-\alpha$, we obtain
$$\chi^2_{\frac{\alpha}{2}}(2n) \leqq \frac{2n\overline{x}}{\lambda} \leqq \chi^2_{1-\frac{\alpha}{2}}(2n).$$

By equivalent transformations we obtain the following $100(1-\alpha)\%$ two-sided confidence interval for the parameter λ

$$\lambda \in \left\langle \frac{2n\overline{x}}{\chi^2_{1-\frac{\alpha}{2}}(2n)}; \frac{2n\overline{x}}{\chi^2_{\frac{\alpha}{2}}(2n)} \right\rangle. \tag{93}$$

We proceed similarly when determining the left-sided and right-sided confidence intervals with probability $1-\alpha$. Thus, the $100(1-\alpha)\%$ left-sided confidence interval for the parameter λ has the form

$$\lambda \in \left\langle \frac{2n\overline{x}}{\chi^2_{1-\alpha}(2n)}; \infty \right) \tag{94}$$

and the $100(1-\alpha)\%$ right-sided confidence interval for the parameter λ has the form

$$\lambda \in \left(0; \frac{2n\overline{x}}{\chi^2_{\alpha}(2n)} \right\rangle. \tag{95}$$

Example 4.5 *For 100 randomly selected passengers, the waiting time for a bus [in min.] was recorded. The results are given in the following table:*

x_j	*0,5*	*2,5*	*4,5*	*6,5*	*8,5*
n_j	*60*	*20*	*11*	*5*	*4*

We assume that the waiting time has an exponential distribution with parameter λ. Determine

a) the 95% two-sided confidence interval for the parameter λ,

b) the 99% left-sided confidence interval for the parameter λ,

c) the 99% right-sided confidence interval for the parameter λ.

Solution: Based on the problem statement, it is clear that we are to determine the confidence interval for "estimating the parameter of the exponential distribution". First we need a suitable point estimate of the mean μ, which is the sample arithmetic mean $\overline{x}$ calculated according to relation (72)

$$\overline{x} = \frac{1}{100}\big(0,5 \cdot 60 + 2,5 \cdot 20 + 4,5 \cdot 11 + 6,5 \cdot 5 + 8,5 \cdot 4\big) = 1,96,$$

so the point estimate of the parameter λ is $1,96$.

a) To determine the two-sided confidence interval we use relation (93). In it we need to determine $\alpha = 0,05$ and the χ^2-distribution quantiles $\chi^2_{\frac{\alpha}{2}}(2n) = \chi^2_{0,025}(200) = 162,73$, $\chi^2_{1-\frac{\alpha}{2}}(2n) = \chi^2_{0,975}(200) = 241,06$ (Table 32). Substituting into relation (93) we obtain the 95% two-sided confidence interval for the parameter λ

$$\lambda \in \left\langle \frac{200 \cdot 1,96}{241,06}; \frac{200 \cdot 1,96}{162,73} \right\rangle \Rightarrow \langle 1,626; 2,409 \rangle .$$

b) To determine the left-sided confidence interval we use relation (94). In it we need to determine $\alpha = 0,01$ and the χ^2-distribution quantile $\chi^2_{1-\alpha}(2n) = \chi^2_{0,99}(200) = 249,45$ (Table 32). Substituting into relation (94) we obtain the 99% left-sided confidence interval for the parameter λ

$$\lambda \in \left\langle \frac{200 \cdot 1,96}{249,45}; \infty \right) \Rightarrow \lambda \in \langle 1,572; \infty) .$$

c) To determine the right-sided confidence interval we use relation (95). In it we need to determine $\alpha = 0,01$ and the χ^2-distribution quantile $\chi^2_{\alpha}(2n) = \chi^2_{0,99}(200) = 156,43$ (Table 32). Substituting into relation (95) we obtain the 99% right-sided confidence interval for the parameter λ

$$\lambda \in \left(0; \frac{200 \cdot 1,96}{156,43} \right\rangle \Rightarrow \lambda \in (0; 2,506\rangle .$$