REZIDUA



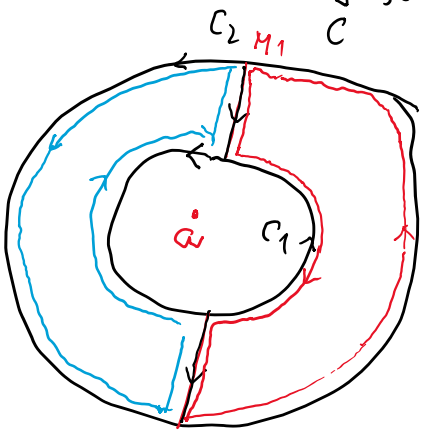
C - uzavretá $\circlearrowleft$, $f(z)$ analyt. na C aj vnútri s výnimkou a - pól I. rádu

$$\int_C f(z)\,dz = 2\pi i \operatorname{res} f(a)$$

?

$$\int_{C_1} = \int_{C_2}$$

$M_1 M_2$ - úsečky, ktoré spájajú C_1, C_2



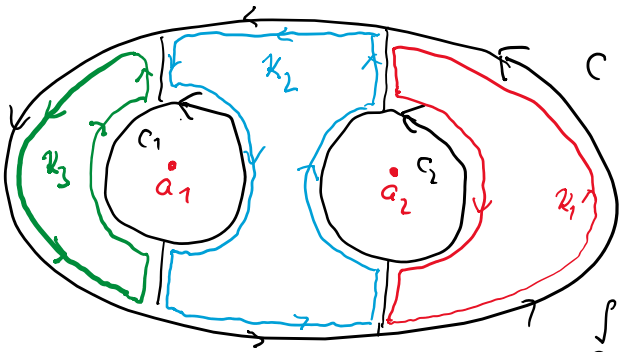
K_1 - červené → $\overset{-}{\text{C}}\text{onč}\ C_2$, M_1, $\overset{-}{\text{C}}\text{onč}\ -C_1$, M_2

K_2 → $\overset{-}{\text{C}}\text{onč}\ C_2$, $-M_2$, $\overset{-}{\text{C}}\text{onč}\ -C_1$, $-M_1$

$$\int_{K_1} f(z)\,dz = 0 \qquad \int_{K_2} f(z)\,dz = 0$$

$$0 = \int_{K_1} + \int_{K_2} = \int_C f(z)\,dz - \int_{C_1} f(z)\,dz$$

Zovšeobecnenie

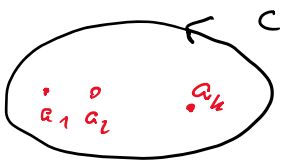


$$0 = \int_{K_1} + \int_{K_2} + \int_{K_3}$$

$$0 = \int_C f(z)\,dz - \int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$\int_C f(z)\,dz = 2\pi i \operatorname{res} f(a_1) + 2\pi i \operatorname{res} f(a_2)$$

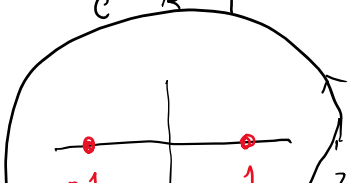


Veta: Nech f je analyt. na C aj jej vnútri s výnimkou bodov $a_1, a_2, \dots, a_k$ - póly 1. rádu f. Tak

$$\int_C f(z)\,dz = 2\pi i \left(\operatorname{res} f(a_1) + \operatorname{res} f(a_2) + \dots + \operatorname{res} f(a_k) \right)$$

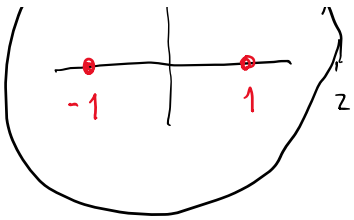
Výp.

$$\int_C \frac{z^3}{z^2-1}\,dz \qquad C: |z| = 2 \qquad 1, -1 \in \text{vn } C$$



$$= 2\pi i \left(\operatorname{res} f(1) + \operatorname{res} f(-1) \right) = 2\pi i$$

$$\left(\frac{z^3}{\quad} \right) \qquad 1$$



$$\operatorname{res} f(1) = \lim_{z\to 1}\left((z-1)\frac{z^3}{(z-1)(z+1)}\right) = \frac{1}{2}$$

$$\operatorname{res} f(-1) = \lim_{z\to -1}\left((z+1)\frac{z^3}{(z-1)(z+1)}\right) = \frac{-1}{-2} = \frac{1}{2}$$

a-pol 1. řádu — analyt.

I. $\int_C f(z)\,dz = \int_C \frac{g(z)}{z-a}\,dz = 2\pi i\, g(a) = 2\pi i \operatorname{res} f(a)$

derivuj podle a

$\to \left(\frac{1}{z-a}\right)' = \left((z-a)^{-1}\right)' = -1(z-a)^{-2}\cdot(-1) = \frac{1}{(z-a)^2}$

a-pól (2.) řádu

II. $\int_C f(z)\,dz = \int_C \frac{g(z)}{(z-a)^2}\,dz = 2\pi i\, g'(a) = 2\pi i\, g'(z)\Big|_{z=a} = 2\pi i \lim_{z\to a}\left((z-a)^2 f(z)\right)'$

$= 2\pi i \operatorname{res} f(a)$

— derivuj podle a

Máme rovnější zápis pomocí REZIDUUM

III. $2\int_C f(z)\,dz = \frac{2}{}\int_C \frac{g(z)}{(z-a)^3}\,dz = 2\pi i\, g''(a) = 2\pi i\, g''(z)\Big|_{z=a} = 2\pi i \lim_{z\to a}\left((z-a)^3 f(z)\right)''$

FINÁLNÝ VZOREC PRE POL 3. RÁDU

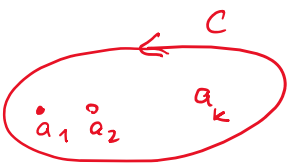
$$\int_C f(z)\,dz = 2\pi i\left[\lim_{z\to a}\frac{1}{2}\left((z-a)^3 f(z)\right)''\right]$$

res f(a) pre pol (3.) rádu

a - pol n-tého rádu fcie f $\left[\text{t.j. } f(z) = \frac{g(z)}{(z-a)^n}\right]$ g(z) analyt. v a

$$\operatorname{res} f(a) = \frac{1}{(n-1)!}\lim_{z\to a}\left((z-a)^n f(z)\right)^{(n-1)}$$

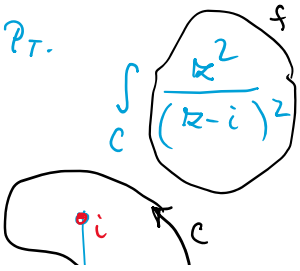
$(n-1)$ $n-1$ ↳ Derive



Ak $f(z)$ je analytická na C aj vnútri C s výnimkou bodov $a_1 \dots a_k$ → póly $f(z)$. Potom

$$\int_C f(z)\,dz = 2\pi i\left(\operatorname{res} f(a_1) + \dots + \operatorname{res} f(a_k)\right)$$

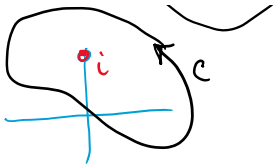
Př.



$$\int_C \frac{z^2}{(z-i)^2}\,dz = 2\pi i \operatorname{res} f(i) = 4\pi i^2 = 4\pi$$

i - pól 2. rádu

$$\operatorname{res} f(i) \lim\left((z-i)^2\frac{z^2}{(z-i)^2}\right)' = \lim_{z\to i} 2z = 2i$$



$$\operatorname{res} f(i) \lim_{z\to i}\left((z-i)^2\frac{z^2}{(z-i)^2}\right)' = \lim_{z\to i} 2z = 2i$$

? $\cos z$, $\sin z$?

$\sin 0 = 0$

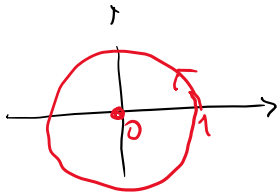
$\cos 0 = 1$

$$(\cos z)' = -\sin z$$
$$(\sin z)' = \cos z$$

Pre výpočet reziduí FAKTY

Př. $\int_C \frac{\sin z}{z^2}\,dz$: $C: |z|=1$

0 - pól 2. rádu



$$\operatorname{res} f(0) = \lim_{z\to 0}\left(z^2\frac{\sin z}{z^2}\right)' = \lim_{z\to 0}\cos z = 1$$

$$2\pi i \operatorname{res} f(0) = 2\pi i$$

$$f(z) = \frac{z}{z^2-1} = \frac{z}{(z-1)(z+1)} = \frac{f_1(z)}{f_2(z)}$$ 1 - pól 1. rádu

$$f(z) = \frac{f_1(z)}{f_2(z)}$$ a - pól 1. rádu $\quad f_2(a)=0 \quad f_2'(a)\neq 0$

$$\operatorname{res} f(a) = \lim_{z\to a}(z-a)\frac{f_1(z)}{f_2(z)} = \lim_{z\to a}\frac{f_1(z)}{\frac{f_2(z)-f_2(a)}{z-a}} = \frac{f_1(a)}{f_2'(a)}$$

a.z. a je pól 1. rádu fce $f(z) = \frac{f_1(z)}{f_2(z)}$, tak $\operatorname{res} f(a) = \frac{f_1(a)}{f_2'(a)}$

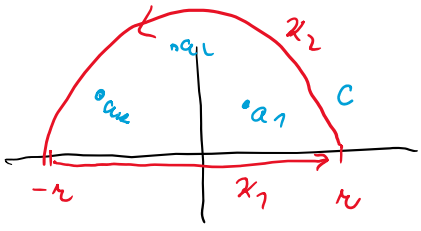
Př. $f(z) = \frac{z}{z^2-1}$ 1 - pól 1. rádu $\operatorname{res} f(a) = \frac{z}{2z}\Big|_1 = \frac{1}{2}$

JEDNO PREPOJENIE KOMPL. A REÁL. INTEGRÁLOV

ÚSTUP

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$$ $\quad st\,P \le st\,Q - 2$ $\quad Q$ - nemá reál. korene

MEDZIKROK -> VYROBÍME KOMP. INT



z_1 - úsečka od -r po r na reál. osi

z_2 - polkružnica s polomerom r

$C = z_1 \cup z_2$ uzavretá krivka

$st\,P \le st\,Q - 2$

C - hrubé horná kompl. polrovina

$$\lim_{r\to\infty}\int_C \frac{P(z)}{Q(z)}\,dz = \lim_{r\to\infty}\left(\int_{z_1}\frac{P(z)}{Q(z)}\,dz + \int_{z_2}\frac{P(z)}{Q(z)}\,dz\right)$$

$\int_{z_2} \to 0$

C - hrdle horna polokruh

$$\lim_{r\to\infty} \int_C \frac{P(z)}{Q(z)}\,dz = \lim_{r\to\infty} \left(\int_{x_1}^{r} \frac{P(x)}{Q(x)}\,dx + \int_{k_2}^{r} \frac{P(z)}{Q(z)}\,dz \right)$$

$$= 2\pi i \sum_{\substack{i=1 \\ \mathrm{Im}\, a_k > 0}}^{k} \operatorname{res} f(a_k)$$

= součet přes póly ležící v horní polorovině

$$\lim_{r\to\infty} \int_{-r}^{r} \frac{P(x)}{Q(x)}\,dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

VSTUP

Vš $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$. Kdeže $f(z) = \frac{P(z)}{Q(z)}$



$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \left(\operatorname{res} f(a_k) + \dots + \operatorname{res} f(a_l) \right)$$

a_k - pól $f(z) = \frac{P(z)}{Q(z)}$ s imag. složkou > 0

Př.

Problém 1. nevlastní integr.

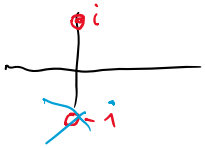
2. asi najít Primit. Fcii $\int \frac{1}{(1+x^2)^2}$ → PP

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx = 2\pi i \operatorname{res} f(i) = \frac{2\pi i}{4i} = \frac{\pi}{2}$$

$$f(z) = \frac{1}{(1+z^2)^2} = \frac{1}{(z+i)^2 (z-i)^2}$$

$$\operatorname{res} f(i) = \lim_{z\to i} \left((z-i)^2 \frac{1}{(z-i)^2 (z+i)^2} \right)' = \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{8(-i)}$$

$$= \frac{1}{4i}$$



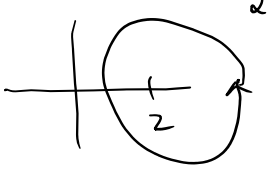
i - pol 2. řádu

$$(e^z)' = e^z$$

Př.

$$\int_C \frac{\sin z\, e^z}{(z-2)^2}\,dz = 2\pi i \operatorname{res} f(2)$$

2 - pol 2. řádu



$$\operatorname{res} f(2) = \lim_{z\to 2} \left(\sin z\, e^z \right)' = \lim_{\substack{z\to 2 \\ \text{real}}} \left(\cos z\, e^z + \sin z\, e^z \right)$$

$$= \cos 2 e^2 + \sin 2 e^2$$