The value of the test statistic $\chi^2 = 196$ does not fall into the critical region $K_{0,01}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,01$, the average waiting time is not smaller than the waiting time given by the carrier.

## 5.3 Two-sample parametric tests of parameters of the normal distribution

Consider two random samples $V_{n_1}$ and $V_{n_2}$. The random sample $V_{n_1}$ has size $n_1$ and comes from a population with a normal distribution with mean value $\mu_1$ and standard deviation $\sigma_1$. We write $V_{n_1} \sim N(\mu_1, \sigma_1^2)$. The random sample $V_{n_2}$ has size $n_2$ and comes from a population with a normal distribution with mean value $\mu_2$ and standard deviation $\sigma_2$. We write $V_{n_2} \sim N(\mu_2, \sigma_2^2)$.

When testing we must take into account whether the random samples come from dependent or independent populations. In the case of *independent populations* we assume that a sample from one population does not depend on the sample from the other population. Conversely, for *dependent populations* we assume that a sample from one population depends on the sample from the other population.

If we do not reject the null hypothesis about the equality of the given parameters, there is no statistically significant difference between the tested parameters. The general procedure for choosing a test of mean values for two samples is shown in Fig. 23.

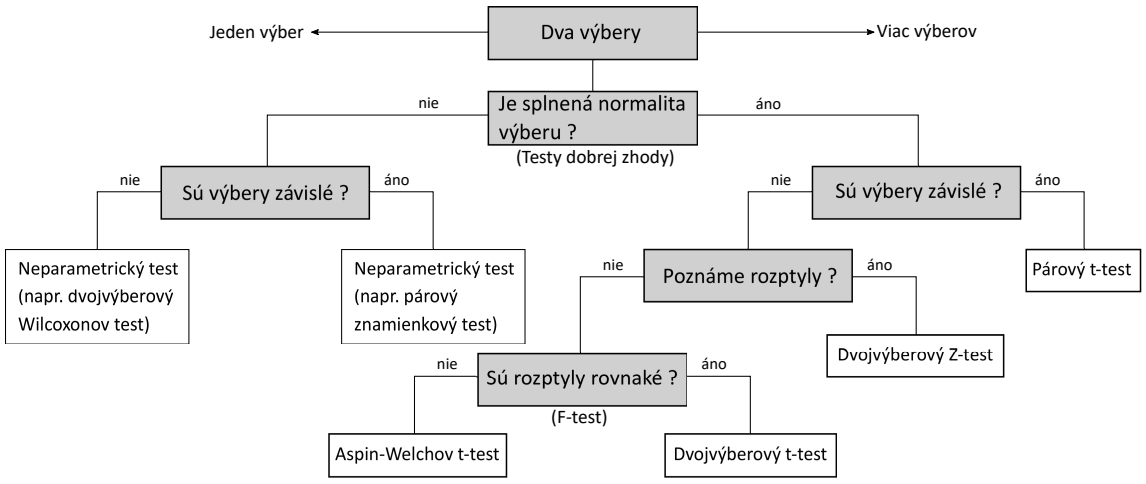



**Figure 23:** Procedure for choosing a test of mean values for two samples

### 5.3.1 Test for equality of variances of two independent samples (Fisher's F-test)

Assume that we have two independent random samples $V_{n_1}$ and $V_{n_2}$, where $V_{n_1} \sim N(\mu_1, \sigma_1^2)$ and $V_{n_2} \sim N(\mu_2, \sigma_2^2)$. The random sample $V_{n_1}$ has size $n_1$ and the random sample $V_{n_2}$ has size $n_2$.

1. Hypotheses:

$$H_0 : \sigma_1^2 = \sigma_2^2,$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2 \quad (\text{resp. } \sigma_1^2 < \sigma_2^2, \text{ resp. } \sigma_1^2 > \sigma_2^2).$$

2. The test statistic is

$$F = \frac{s_1^2}{s_2^2} \sim F(n_1 - 1, n_2 - 1), \tag{101}$$

where $s_1^2$, $s_2^2$ are the sample variances, for which

$$s_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_{1i} - \overline{x}_1)^2 , \qquad s_2^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (x_{2i} - \overline{x}_2)^2 ,$$

where $\overline{x}_1$ and $\overline{x}_2$ are the sample arithmetic means of the given random samples.

3. The critical region $K_\alpha$ for various alternatives is given in the following table:

| $H_0$ | $H_1$ | $K_\alpha$ |
|---|---|---|
| $\sigma_1^2 = \sigma_2^2$ | $\sigma_1^2 \neq \sigma_2^2$ | $\left(0; F_{\frac{\alpha}{2}}(n_1 - 1, n_2 - 1)\right) \cup \left(F_{1-\frac{\alpha}{2}}(n_1 - 1, n_2 - 1); \infty\right)$ |
| $\sigma_1^2 = \sigma_2^2$ | $\sigma_1^2 > \sigma_2^2$ | $(F_{1-\alpha}(n_1 - 1, n_2 - 1); \infty)$ |
| $\sigma_1^2 = \sigma_2^2$ | $\sigma_1^2 < \sigma_2^2$ | $(0; F_{\alpha}(n_1 - 1, n_2 - 1))$ |

### 5.3.2 Test for equality of mean values of two independent samples

**Two-sample Z-test**

We use the two-sample Z-test if we know the population variances $\sigma_1^2$, $\sigma_2^2$.

1. Hypotheses:

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 \neq \mu_2 \quad (\text{resp. } \mu_1 < \mu_2, \text{ resp. } \mu_1 > \mu_2).$$

2. The test statistic is

$$Z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1), \tag{102}$$

where $\overline{x}_1$ and $\overline{x}_2$ are the sample arithmetic means of the given random samples.

3. The critical region $K_\alpha$ for various alternatives is given in the following table:

| $H_0$ | $H_1$ | $K_\alpha$ |
|---|---|---|
| $\mu_1 = \mu_2$ | $\mu_1 \neq \mu_2$ | $\left(-\infty; -u_{1-\frac{\alpha}{2}}\right) \cup \left(u_{1-\frac{\alpha}{2}}; \infty\right)$ |
| $\mu_1 = \mu_2$ | $\mu_1 > \mu_2$ | $(u_{1-\alpha}; \infty)$ |
| $\mu_1 = \mu_2$ | $\mu_1 < \mu_2$ | $(-\infty; -u_{1-\alpha})$ |

**Example 5.6** *Two types of protective coating were used to coat 24 objects. The first coating was used on 10 objects and the second on 14 objects. After a certain period, the degree of wear was determined on each object. We found that the average degree of wear for the first type of coating was 62, with $\sigma_1^2$ being 16, and the average degree of wear for the second type of coating was 60, with $\sigma_2^2$ being 15. We assume a normal distribution for both populations from which the random samples were drawn.*

*a) Let us verify the assumption of equal effectiveness of both types of protective coatings at the significance level $\alpha$=0.05.*

*b) Let us verify the assumption of higher effectiveness (lower degree of wear) of the first type of protective coating at the significance level $\alpha$=0.01.*

*c) Let us verify the assumption of lower effectiveness (higher degree of wear) of the first type of protective coating at the significance level $\alpha$=0.1.*

**Solution:** We are testing the equality of mean values of two independent samples. For testing we use the two-sample Z-test (we know the population variances). The size of the first random sample is $n_1 = 10$ and the size of the second random sample is $n_2 = 14$.

a) We will test

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 \neq \mu_2.$$

The null hypothesis states that both protective coatings have the same effectiveness. Against the alternative hypothesis, which states that the protective coatings do not have the same effectiveness. From the assignment it follows that the sample arithmetic means are $\overline{x}_1 = 62$,

$\overline{x}_2 = 60$ and the population variances are $\sigma_1^2 = 16$, $\sigma_2^2 = 15$. The value of the test statistic $Z$ is, according to relation (102)

$$Z = \frac{62 - 60}{\sqrt{\frac{16}{10} + \frac{15}{14}}} = 1,2237.$$

The quantile of the standard normal distribution is $u_{1-\frac{\alpha}{2}} = u_{0,975} = 1,96$ (Table 28). The critical region is

$$K_{0,05} = (-\infty; -1,96) \cup (1,96; \infty) \Rightarrow Z \notin K_{0,05}.$$

The value of the test statistic $Z = 1,2237$ does not fall into the critical region $K_{0,05}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,05$, the protective coatings have the same effectiveness.

b) We will test

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 < \mu_2.$$

The null hypothesis states that both protective coatings have the same effectiveness. Against the alternative hypothesis, which states that the first type of protective coating has a lower degree of wear, i.e. higher effectiveness. We calculate the value of the test statistic $Z$ in the same way as in case a), and thus $Z = 1,2237$. The quantile of the standard normal distribution is $u_{1-\alpha} = u_{0,99} = 2,3264$ (Table 28). The critical region is

$$K_{0,01} = (-\infty; -2,3264) \Rightarrow Z \notin K_{0,01}.$$

The value of the test statistic $Z = 1,2237$ does not fall into the critical region $K_{0,01}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,01$, the first type of protective coating does not have higher effectiveness than the second type of coating.

c) We will test

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 > \mu_2.$$

The null hypothesis states that both protective coatings have the same effectiveness. Against the alternative hypothesis, which states that the first type of protective coating has a higher degree of wear, i.e. lower effectiveness. We calculate the value of the test statistic $Z$ in the same way as in case a), and thus $Z = 1,2237$. The quantile of the standard normal distribution is $u_{1-\alpha} = u_{0,9} = 1,2816$ (Table 28). The critical region is

$$K_{0,1} = (1,2816; \infty) \Rightarrow Z \notin K_{0,1}.$$

The value of the test statistic $Z = 1,2237$ does not fall into the critical region $K_{0,1}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,1$, the first type of protective coating does not have lower effectiveness than the second type of coating.

**Two-sample t-test**

We use the two-sample t-test if we do not know the population variances $\sigma_1^2$, $\sigma_2^2$, but we can assume that $\sigma_1^2 = \sigma_2^2$.

1. Hypotheses:

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 \neq \mu_2 \quad (\text{resp. } \mu_1 < \mu_2, \text{ resp. } \mu_1 > \mu_2).$$

2. The test statistic is

$$t = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{(n_1-1)s_1^2 + (n_2-1)s_2^2}} \cdot \sqrt{\frac{n_1 n_2 (n_1+n_2-2)}{n_1+n_2}} \sim t(n_1+n_2-2), \tag{103}$$

where $s_1^2$, $s_2^2$ are the sample variances, for which

$$s_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_{1i}-\overline{x}_1)^2, \qquad s_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}(x_{2i}-\overline{x}_2)^2,$$

where $\overline{x}_1$ and $\overline{x}_2$ are the sample arithmetic means of the given random samples.

3. The critical region $K_\alpha$ for various alternatives is given in the following table:

| $H_0$ | $H_1$ | $K_\alpha$ |
|---|---|---|
| $\mu_1 = \mu_2$ | $\mu_1 \neq \mu_2$ | $\left(-\infty; -t_{1-\frac{\alpha}{2}}(n_1+n_2-2)\right) \cup \left(t_{1-\frac{\alpha}{2}}(n_1+n_2-2); \infty\right)$ |
| $\mu_1 = \mu_2$ | $\mu_1 > \mu_2$ | $\left(t_{1-\alpha}(n_1+n_2-2); \infty\right)$ |
| $\mu_1 = \mu_2$ | $\mu_1 < \mu_2$ | $\left(-\infty; -t_{1-\alpha}(n_1+n_2-2)\right)$ |

**Aspin-Welch test**

We use the Aspin-Welch test if we do not know the population variances $\sigma_1^2$, $\sigma_2^2$, but we can assume that $\sigma_1^2 \neq \sigma_2^2$.

1. Hypotheses:

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 \neq \mu_2 \quad (\text{resp. } \mu_1 < \mu_2, \text{ resp. } \mu_1 > \mu_2).$$

2. The test statistic is

$$t = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim t(m), \tag{104}$$

where

$$m = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}}, \tag{105}$$

where $s_1^2$, $s_2^2$ are the sample variances, for which

$$s_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_{1i} - \overline{x}_1)^2, \qquad s_2^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (x_{2i} - \overline{x}_2)^2,$$

where $\overline{x}_1$ and $\overline{x}_2$ are the sample arithmetic means of the given random samples.

3. The critical region $K_\alpha$ for various alternatives is given in the following table:

| $H_0$ | $H_1$ | $K_\alpha$ |
|---|---|---|
| $\mu_1 = \mu_2$ | $\mu_1 \neq \mu_2$ | $\left(-\infty; -t_{1-\frac{\alpha}{2}}(m)\right) \cup \left(t_{1-\frac{\alpha}{2}}(m); \infty\right)$ |
| $\mu_1 = \mu_2$ | $\mu_1 > \mu_2$ | $(t_{1-\alpha}(m); \infty)$ |
| $\mu_1 = \mu_2$ | $\mu_1 < \mu_2$ | $(-\infty; -t_{1-\alpha}(m))$ |

**Example 5.7** *Two types of protective coating were used to coat 24 objects. The first coating was used on 10 objects and the second on 14 objects. After a certain period, the degree of wear was determined on each object. We found that the average degree of wear for the first type of coating was 62, with $s_1^2$ being 16, and the average degree of wear for the second type of coating was 60, with $s_2^2$ being 15. We assume a normal distribution for both populations from which the random samples were drawn.*

*a) Let us verify the assumption of equal effectiveness of both types of protective coatings at the significance level $\alpha$=0.05.*

*b) Let us verify the assumption of higher effectiveness (lower degree of wear) of the first type of protective coating at the significance level $\alpha$=0.01.*

*c) Let us verify the assumption of lower effectiveness (higher degree of wear) of the first type of protective coating at the significance level $\alpha$=0.1.*

**Solution:** We are testing the equality of mean values of two independent samples, where we do not know the population variances. The size of the first random sample is $n_1 = 10$ and the size of the second random sample is $n_2 = 14$. First we must verify whether we can assume that the variances of both populations are equal or different. Based on that we will decide which of the tests (two-sample t-test or Aspin-Welch test) we will use to test the equality of the mean values of the two independent samples.

a) To verify whether the variances of both populations are equal or different, we use the test for equality of variances of two independent samples (Fisher's F-test). We will test

$$H_0 : \sigma_1^2 = \sigma_2^2,$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2.$$

The null hypothesis states that the assumption of equal variances is satisfied. Against the alternative hypothesis, which states that the assumption of equal variances is not satisfied. The value of the test statistic $F$ is, according to relation (101)

$$F = \frac{16}{15} = 1,067.$$

The quantiles of Fisher's F-distribution are $F_{1-\frac{\alpha}{2}}(n_1 - 1, n_2 - 1) = F_{0,975}(9, 13) = 3,312$ and $F_{\frac{\alpha}{2}}(n_1 - 1, n_2 - 1) = F_{0,025}(9, 13) = 0,2611$ (Table 38). The critical region is

$$K_{0,05} = (0; 0,2611) \cup (3,312; \infty) \Rightarrow F \notin K_{0,05}.$$

The value of the test statistic $F = 1,067$ does not fall into the critical region $K_{0,05}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,05$, the differences between the variances of both populations are not statistically significant. It follows that we will use the two-sample t-test to test the equality of the mean values of the two independent samples.

We will test

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 \neq \mu_2.$$

The null hypothesis states that both protective coatings have the same effectiveness. Against the alternative hypothesis, which states that the protective coatings do not have the same effectiveness. From the assignment it follows that the sample arithmetic means are $\overline{x}_1 = 62$, $\overline{x}_2 = 60$ and the sample variances are $s_1^2 = 16$, $s_2^2 = 15$. The value of the test statistic $t$ is, according to relation (103)

$$t = \frac{62 - 60}{\sqrt{(10 - 1) \cdot 16 + (14 - 1) \cdot 15}} \cdot \sqrt{\frac{10 \cdot 14 \cdot (10 + 14 - 2)}{10 + 14}} = 1,2306.$$

The quantile of the Student t-distribution is $t_{1-\frac{\alpha}{2}}(n_1+n_2-2) = t_{0,975}(22) = 2,0739$ (Table 29). The critical region is

$$K_{0,05} = (-\infty; -2,0739) \cup (2,0739; \infty) \Rightarrow t \notin K_{0,05}.$$

The value of the test statistic $t = 1,2306$ does not fall into the critical region $K_{0,05}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,05$, the protective coatings have the same effectiveness.

b) To verify whether the variances of both populations are equal or different, we use the test for equality of variances of two independent samples (Fisher's F-test). We will test

$$H_0 : \sigma_1^2 = \sigma_2^2,$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2.$$

The null hypothesis states that the assumption of equal variances is satisfied. Against the alternative hypothesis, which states that the assumption of equal variances is not satisfied. We calculate the value of the test statistic $F$ in the same way as in case a), and thus $F = 1,067$. The quantiles of Fisher's F-distribution are $F_{1-\frac{\alpha}{2}}(n_1-1, n_2-1) = F_{0,995}(9,13) = 4,9351$ and $F_{\frac{\alpha}{2}}(n_1-1, n_2-1) = F_{0,005}(9,13) = 0,1625$ (Table 42). The critical region is

$$K_{0,01} = (0; 0,1625) \cup (4,9351; \infty) \Rightarrow F \notin K_{0,01}.$$

The value of the test statistic $F = 1,067$ does not fall into the critical region $K_{0,01}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,01$, the differences between the variances of both populations are not statistically significant. It follows that we will use the two-sample t-test to test the equality of the mean values of the two independent samples.

We will test

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 < \mu_2.$$

The null hypothesis states that both protective coatings have the same effectiveness. Against the alternative hypothesis, which states that the first type of protective coating has a lower degree of wear, i.e. higher effectiveness. We calculate the value of the test statistic $t$ in the same way as in case a), and thus $t = 1,2306$. The quantile of the Student t-distribution is $t_{1-\alpha}(n_1+n_2-2) = t_{0,999}(22) = 2,5083$ (Table 29). The critical region is

$$K_{0,01} = (-\infty; -2,5083) \Rightarrow t \notin K_{0,01}.$$

The value of the test statistic $t = 1,2306$ does not fall into the critical region $K_{0,01}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,01$, the first type of protective coating does not have higher effectiveness than the second type of protective coating.

c) To verify whether the variances of both populations are equal or different, we use the test for equality of variances of two independent samples (Fisher's F-test). We will test

$$H_0 : \sigma_1^2 = \sigma_2^2,$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2.$$

The null hypothesis states that the assumption of equal variances is satisfied. Against the alternative hypothesis, which states that the assumption of equal variances is not satisfied. We calculate the value of the test statistic $F$ in the same way as in case a), and thus $F = 1,067$. The quantiles of Fisher's F-distribution are $F_{1-\frac{\alpha}{2}}(n_1 - 1, n_2 - 1) = F_{0,95}(9, 13) = 2,7144$ and $F_{\frac{\alpha}{2}}(n_1 - 1, n_2 - 1) = F_{0,05}(9, 13) = 0,3281$ (Table 34). The critical region is

$$K_{0,1} = (0; 0,3281) \cup (2,7144; \infty) \Rightarrow F \notin K_{0,1}.$$

The value of the test statistic $F = 1,067$ does not fall into the critical region $K_{0,1}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,1$, the differences between the variances of both populations are not statistically significant. It follows that we will use the two-sample t-test to test the equality of the mean values of the two independent samples.

We will test

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 > \mu_2.$$

The null hypothesis states that both protective coatings have the same effectiveness. Against the alternative hypothesis, which states that the first type of protective coating has a higher degree of wear, i.e. lower effectiveness. We calculate the value of the test statistic $t$ in the same way as in case a), and thus $t = 1,2306$. The quantile of the Student t-distribution is $t_{1-\alpha}(n_1 + n_2 - 2) = t_{0,9}(22) = 1,3212$ (Table 29). The critical region is

$$K_{0,1} = (1,3212; \infty) \Rightarrow t \notin K_{0,1}.$$

The value of the test statistic $t = 1,2306$ does not fall into the critical region $K_{0,1}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,1$, the first type of protective coating does not have lower effectiveness than the second type of

protective coating.

**Example 5.8** *Two types of protective coating were used to coat 24 objects. The first coating was used on 10 objects and the second on 14 objects. After a certain period, the degree of wear was determined on each object. We found that the average degree of wear for the first type of coating was 62, with $s_1^2$ being 4, and the average degree of wear for the second type of coating was 60, with $s_2^2$ being 16. We assume a normal distribution for both populations from which the random samples were drawn.*

*a) Let us verify the assumption of equal effectiveness of both types of protective coatings at the significance level $\alpha$=0.05.*

*b) Let us verify the assumption of higher effectiveness (lower degree of wear) of the first type of protective coating at the significance level $\alpha$=0.1.*

*c) Let us verify the assumption of lower effectiveness (higher degree of wear) of the first type of protective coating at the significance level $\alpha$=0.05.*

**Solution:** We are testing the equality of mean values of two independent samples, where we do not know the population variances. The size of the first random sample is $n_1 = 10$ and the size of the second random sample is $n_2 = 14$. As in the previous example, we must verify whether we can assume that the variances of both populations are equal or different.

a) To verify whether the variances of both populations are equal or different, we use the test for equality of variances of two independent samples (Fisher's F-test). We will test

$$H_0 : \sigma_1^2 = \sigma_2^2,$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2.$$

The null hypothesis states that the assumption of equal variances is satisfied. Against the alternative hypothesis, which states that the assumption of equal variances is not satisfied. We determine the value of the test statistic $F$ using relation (101)

$$F = \frac{4}{16} = 0,25.$$

The quantiles of Fisher's F-distribution are $F_{1-\frac{\alpha}{2}}(n_1 - 1, n_2 - 1) = F_{0,975}(9, 13) = 3,312$ and $F_{\frac{\alpha}{2}}(n_1 - 1, n_2 - 1) = F_{0,025}(9, 13) = 0,2611$ (Table 38). The critical region is

$$K_{0,05} = (0; 0,2611) \cup (3,312; \infty) \Rightarrow F \in K_{0,05}.$$

The value of the test statistic $F = 0,25$ falls into the critical region $K_{0,05}$, therefore we reject the null hypothesis $H_0$ and accept the alternative hypothesis $H_1$. We can assume that, at the significance level $\alpha = 0,05$, the differences between the variances of both populations are statistically significant. It follows that we will use the Aspin-Welch test to test the equality of the mean values of the two independent samples.

We will test

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 \neq \mu_2.$$

The null hypothesis states that both protective coatings have the same effectiveness. Against the alternative hypothesis, which states that the protective coatings do not have the same effectiveness. From the assignment it follows that the sample arithmetic means are $\overline{x}_1 = 62$, $\overline{x}_2 = 60$ and the sample variances are $s_1^2 = 4$, $s_2^2 = 16$. We determine the value of the test statistic $t$ using relation (104)

$$t = \frac{62 - 60}{\sqrt{\frac{4}{10} + \frac{16}{14}}} = 1,6102.$$

The quantile of the Student t-distribution is $t_{1-\frac{\alpha}{2}}(m) = t_{0,975}(20) = 2,0851$ (Table 29), where $m$ is calculated using relation (105)

$$m = \frac{\left(\frac{4}{10} + \frac{16}{14}\right)^2}{\frac{\left(\frac{4}{10}\right)^2}{10-1} + \frac{\left(\frac{16}{14}\right)^2}{14-1}} = 20,1305 \approx 20.$$

The critical region is

$$K_{0,05} = (-\infty; -2,0851) \cup (2,0851; \infty) \Rightarrow t \notin K_{0,05}.$$

The value of the test statistic $t = 1,6102$ does not fall into the critical region $K_{0,05}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,05$, the protective coatings have the same effectiveness.

b) To verify whether the variances of both populations are equal or different, we use the test for equality of variances of two independent samples (Fisher's F-test). We will test

$$H_0 : \sigma_1^2 = \sigma_2^2,$$
$$H_1 : \sigma_1^2 \neq \sigma_2^2.$$

The null hypothesis states that the assumption of equal variances is satisfied. Against the alternative hypothesis, which states that the assumption of equal variances is not satisfied.

We calculate the value of the test statistic $F$ in the same way as in case a), and thus $F = 0,25$. The quantiles of Fisher's F-distribution are $F_{1-\frac{\alpha}{2}}(n_1 - 1, n_2 - 1) = F_{0,95}(9, 13) = 2,7144$ and $F_{\frac{\alpha}{2}}(n_1 - 1, n_2 - 1) = F_{0,05}(9, 13) = 0,3281$ (Table 34). The critical region is

$$K_{0,1} = (0; 0,3281) \cup (2,7144; \infty) \Rightarrow F \in K_{0,1}.$$

The value of the test statistic $F = 0,25$ falls into the critical region $K_{0,1}$, therefore we reject the null hypothesis $H_0$ and accept the alternative hypothesis $H_1$. We can assume that, at the significance level $\alpha = 0,1$, the differences between the variances of both populations are statistically significant. It follows that we will use the Aspin-Welch test to test the equality of the mean values of the two independent samples.

We will test

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 < \mu_2.$$

The null hypothesis states that both protective coatings have the same effectiveness. Against the alternative hypothesis, which states that the first type of protective coating has a lower degree of wear, i.e. higher effectiveness. We calculate the value of the test statistic $t$ in the same way as in case a), and thus $t = 1,6102$. The quantile of the Student t-distribution is $t_{1-\alpha}(m) = t_{0,9}(20) = 1,3251$ (Table 29), where $m$ is calculated in the same way as in case a). The critical region is

$$K_{0,1} = (-\infty; -1,3251) \Rightarrow t \notin K_{0,1}.$$

The value of the test statistic $t = 1,6102$ does not fall into the critical region $K_{0,1}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,1$, the protective coatings have the same effectiveness.

c) From task a) we found that, at the significance level $\alpha = 0,05$, we can assume that the differences between the variances of both populations are statistically significant. It follows that we will use the Aspin-Welch test to test the equality of the mean values of the two independent samples.

We will test

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 > \mu_2.$$

The null hypothesis states that both protective coatings have the same effectiveness. Against the alternative hypothesis, which states that the first type of protective coating has a higher degree of wear, i.e. lower effectiveness. We calculate the value of the test statistic $t$ in the

same way as in case a), and thus $t = 1,6102$. The quantile of the Student t-distribution is $t_{1-\alpha}(m) = t_{0,95}(20) = 1,7242$ (Table 29), where $m$ is calculated in the same way as in case a). The critical region is

$$K_{0,05} = (1,7242; \infty) \Rightarrow t \notin K_{0,05}.$$

The value of the test statistic $t = 1,6102$ does not fall into the critical region $K_{0,05}$, therefore we do not reject the null hypothesis $H_0$. We can assume that, at the significance level $\alpha = 0,05$, the protective coatings have the same effectiveness.

### 5.3.3 Test for equality of mean values of two dependent samples

If both random samples $V_{n_1}$ and $V_{n_2}$ come from dependent populations and have the same size (i. e. $n_1 = n_2 = n$), we can use the *paired t-test* (e.g., if we measure some set, object twice: before and after some event).

1. Hypotheses:

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 \neq \mu_2 \quad (\text{resp. } \mu_1 < \mu_2, \text{ resp. } \mu_1 > \mu_2).$$

2. The test statistic is

$$t = \frac{\overline{d}}{s_d}\sqrt{n} \sim t(n-1), \tag{106}$$

where $\overline{d}$ is the sample arithmetic mean of the differences

$$\overline{d} = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{1}{n}\sum_{i=1}^{n} (x_{1i} - x_{2i}) \tag{107}$$

and $s_d^2$ is the sample variance of the differences

$$s_d^2 = \frac{1}{n-1}\sum_{i=1}^{n} \left(d_i - \overline{d}\right)^2. \tag{108}$$

3. The critical region $K_\alpha$ for various alternatives is given in the following table:

| $H_0$ | $H_1$ | $K_\alpha$ |
|---|---|---|
| $\mu_1 = \mu_2$ | $\mu_1 \neq \mu_2$ | $\left(-\infty; -t_{1-\frac{\alpha}{2}}(n-1)\right) \cup \left(t_{1-\frac{\alpha}{2}}(n-1); \infty\right)$ |
| $\mu_1 = \mu_2$ | $\mu_1 > \mu_2$ | $(t_{1-\alpha}(n-1); \infty)$ |
| $\mu_1 = \mu_2$ | $\mu_1 < \mu_2$ | $(-\infty; -t_{1-\alpha}(n-1))$ |

***Remark:*** These relations are used for small sample sizes (similarly as in chapter 4.2), i. e. if $n_1 \leqq 30$, $n_2 \leqq 30$. Otherwise we approximate the Student t-distribution by the standard normal distribution.

**Example 5.9** *We examined 8 samples of a chemical substance. Each sample was analyzed by the titration method and the polarographic method. The results are given in the table. We assume a normal distribution for both populations from which the random samples were drawn. Let us verify the equivalence of both methods at the significance level $\alpha$=0.01.*

| *sample* | *Polarographic method* | *Titration method* |
|---|---|---|
| *1* | *18,60* | *18,58* |
| *2* | *27,60* | *27,37* |
| *3* | *27,50* | *27,27* |
| *4* | *25,00* | *24,64* |
| *5* | *24,50* | *24,10* |
| *6* | *26,80* | *26,33* |
| *7* | *29,70* | *29,33* |
| *8* | *26,63* | *26,50* |

**Solution:** Both random samples come from dependent populations and have the same size $n_1 = n_2 = n$. We can use the paired t-test.

We will test

$$H_0 : \mu_1 = \mu_2,$$

$$H_1 : \mu_1 \neq \mu_2.$$

The null hypothesis states that the methods are equivalent. Against the alternative hypothesis, which states that the methods are not equivalent. We supplement the given table with a column of differences (Table 6).

We determine the value of the test statistic $t$ using relation (106)

$$t = \frac{\overline{d}}{s_d}\sqrt{8},$$

where the sample arithmetic mean of the differences $\overline{d}$ is calculated using relation (107)

$$\overline{d} = \frac{1}{8}(0,02 + 0,23 + 0,23 + 0,36 + 0,40 + 0,47 + 0,37 + 0,13) = 0,2763$$

and the sample variance of the differences $s_d^2$ is calculated using relation (108)

$$s_d^2 = \frac{1}{7}\big[(0,02 - 0,2763)^2 + (0,23 - 0,2763)^2 + (0,23 - 0,2763)^2 +$$

**Table 6**

| sample ($i$) | Polarographic method ($x_{1i}$) | Titration method ($x_{2i}$) | Differences ($d_i = x_{1i} - x_{2i}$) |
|---|---|---|---|
| 1 | $18,60$ | $18,58$ | $0,02$ |
| 2 | $27,60$ | $27,37$ | $0,23$ |
| 3 | $27,50$ | $27,27$ | $0,23$ |
| 4 | $25,00$ | $24,64$ | $0,36$ |
| 5 | $24,50$ | $24,10$ | $0,40$ |
| 6 | $26,80$ | $26,33$ | $0,47$ |
| 7 | $29,70$ | $29,33$ | $0,37$ |
| 8 | $26,63$ | $26,50$ | $0,13$ |

$$+(0,36-0,2763)^2+(0,40-0,2763)^2+(0,47-0,2763)^2+(0,37-0,2763)^2+$$

$$+(0,13-0,2763)^2\big]=0,0229 \Rightarrow s_d=\sqrt{s_d^2}=\sqrt{0,0229}=0,1513.$$

The value of the test statistic is then $t = 5,1684$. The quantile of the Student t-distribution $t_{1-\frac{\alpha}{2}}(n-1) = t_{0,995}(7) = 3,4995$ (Table 29). The critical region is

$$K_{0,01} = (-\infty; -3,4995) \cup (3,4995; \infty) \Rightarrow t \in K_{0,01}.$$

The value of the test statistic $t = 5,1684$ falls into the critical region $K_{0,01}$, therefore we reject the null hypothesis $H_0$ and accept the alternative hypothesis $H_1$. We can assume that, at the significance level $\alpha = 0,01$, the methods used are not equivalent.

## 5.4 Symmetry tests

Symmetry tests belong among the nonparametric tests. There are several such tests, but we will present the sign test and the Wilcoxon two-sample test.

### 5.4.1 Sign test

The *sign test (median test)* is one of the oldest and simplest nonparametric tests, which tests the symmetry of the distribution of a statistical variable about a constant $x_0$ as the median.

Consider a random sample $V_n$ with a continuous distribution with an unknown median $\widetilde{x}$. The sample size is $n$.