4.5 Cvičenia 4

1. Dokážte , že pre funkciu $f(x,y) = 3x^2y - \sqrt{x^6 - y^6}$ platí
$$f(tx,ty) = t^3 f(x,y) \quad \text{pre } t \geq 0.$$

2. Nájdite $f(x)$, ak $f(\frac{x}{y}) = \frac{x\sqrt{x^2+y^2}}{y^2}$, ak $y > 0$. [$f(x) = x\sqrt{x^2+1}$]

3. Nájdite obor definície funkcie f, ak :
   a) $f(x,y) = \frac{1}{x^2+y^2} + \frac{1}{1-x}$, [$x \neq 1; (x,y) \neq (0,0)$]
   b) $f(x,y) = \frac{x^2+3y}{x^2-y^2}$, [$y \neq \pm x$]
   c) $f(x,y) = \frac{e^x+3xy}{9-x^2-y^2}$. [$x^2+y^2 \neq 9$]

4. Nájdite obor definície funkcie f, ak:
   a) $f(x,y) = \sqrt{1-x^2-y^2}$, [$x^2+y^2 \leq 1$]
   b) $f(x,y) = \frac{5x+y}{\sqrt{xy}}$, [$xy > 0$]
   c) $f(x,y) = \frac{3-y}{\sqrt{x+|y|}} + \frac{5}{\sqrt{x-|y|}}$. [$x > |y|$]

5. Nájdite obor definície funkcie f, ak:
   a) $f(x,y,z) = x - \sqrt{1-x^2-y^2-z^2}$, [$x^2+y^2+z^2 \leq 1$]
   b) $f(x,y,z) = z^2 - \sqrt{x^2+y^2-1}$, [$x^2+y^2 \geq 1$]
   c) $f(x,y,z) = \ln(1+x^2+z^2) + z\ln(1-x^2-y^2)$. [$x^2+y^2 < 1$]

6. Zostrojte graf funkcie f, ak:
   a) $f(x,y) = x^2+y^2$ a $|f(x,y)| \leq 4$,
   b) $f(x,y) = x^2$ a $|f(x,y)| \leq 1$, $0 \leq y \leq 2$,
   c) $f(x,y) = \sqrt{1-x^2-y^2}$.

7. Vypočítajte:
   a) $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\sin(xy)^{-1}$, [0]
   b) $\lim\limits_{(x,y)\to(\infty,\infty)} \frac{x+y}{x^2+y^2}$, [0]
   c) $\lim\limits_{(x,y)\to(0,2)} \frac{\sin xy}{x}$, [2]
   d) $\lim\limits_{(x,y)\to(\infty,k)} (1+\frac{y}{x})^x$, [$e^k$]
   e) $\lim\limits_{(x,y)\to(0,0)} \frac{x}{x+y}$, [neexistuje]
   f) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$. [neexistuje]

8. Vypočítajte parciálne derivácie funkcií:
   a) $z = x^3+y^3-3axy$, [$\frac{\partial z}{\partial x} = 3x^2-3ay$, $\frac{\partial z}{\partial y} = 3y^2-3ax$]
   b) $z = \frac{x-y}{x+y}$, [$\frac{\partial z}{\partial x} = \frac{2y}{(x+y)^2}$, $\frac{\partial z}{\partial y} = \frac{-2x}{(x+y)^2}$]
   c) $z = \operatorname{arctg}\frac{y}{x}$, [$\frac{\partial z}{\partial x} = \frac{-y}{x^2+y^2}$, $\frac{\partial z}{\partial y} = \frac{x}{x^2+y^2}$]
   d) $z = x^y$, $x > 0$. [$\frac{\partial z}{\partial x} = yx^{y-1}$, $\frac{\partial z}{\partial y} = x^y\ln x$]

9. Ukážte, že:
   a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2$, ak $z = \ln(x^2+xy+y^2)$,
   b) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy+z$, ak $z = xy+xe^{y/x}$,
   c) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$, ak $u = (x-y)(y-z)(z-x)$.

10. Nájdite úplný diferenciál nasledujúcich funkcií:
a) $z = x^3 + y^3 - 3xy$, $[dz = 3(x^2 - y)dx + 3(y^2 - x)dy]$
b) $z = yx^y$, $[dz = y^2x^{y-1}dx + x^y(1 + y\ln x)dy]$
c) $z = \ln(x^2 + y^2)$, $[dz = 2(xdx + ydy)/(x^2 + y^2)]$
d) $z = \ln(1 + x/y)$. $[dz = (dx - \frac{x}{y}dy)/(x + y)]$

11. Určte približnú hodnotu výrazov:
a) $(1,02)^3 \cdot (0,97)^2$, [1,00]
b) $\sqrt{(4,05)^2 + (2,93)^2}$, [4,998]
c) $\sin 32^0 \cdot \cos 59^0$. [0,273]

12. Nájdite extrémy funkcií:
a) $f(x, y) = (x - 1)^2 + 2y^2$; $[f_{\min} = 0$ v bode $(1; 0)]$
b) $f(x, y) = x^2 + xy + y^2 - 2x - y$; $[f_{\min} = -1$ v bode $(1; 0)]$
c) $f(x, y) = x^3y^2(6 - x - y)$, $x > 0, y > 0$; $[f_{\max} = 108$ v bode $(3; 2)]$
d) $f(x, y) = e^{x-y}(x^2 - 2y^2)$; $[f_{\min} = 6$ v bode $(4; 2)]$
e) $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.
$[f_{\min} = -4/3$ v bode $(-2/3; -1/3; 1)]$

13. Nájdite extrémy funkcií:
a) $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$;
$[f_{\min} = -8$ v bode $(\sqrt{2}; -\sqrt{2})$ a v bode $(-\sqrt{2}; \sqrt{2})]$
b) $f(x, y) = x^2 + xy + y^2 + 1/x + 1/y$ $x > 0, y > 0$;
$[f_{\min} = 3\sqrt[3]{3}$ v bode $(1/\sqrt[3]{3}; 1/\sqrt[3]{3})]$
c) $f(x, y) = e^{-x^2-y^2}(2x^2 + y^2)$.
$[\ f_{\min} = 0$ v bode $(0; 0)$ a $f_{\max} = 2/e$ v bode $(\pm 1; 0)\ ]$

14. Nájdite extrémy funkcií s väzbou:
a) $f(x, y) = xy$, $g(x, y) \equiv x + y - 1 = 0$;
$[f_{\max} = 1/4$ v bode $(1/2; 1/2)]$
b) $f(x, y) = x + 2y$, $g(x, y) \equiv x^2 + y^2 = 5$;
$[f_{\max} = 5$ v bode $(1; 2)$ a $f_{\min} = -5$ v bode $(-1; -2)]$
c) $f(x, y) = x^2 + y^2$, $g(x, y) \equiv 3x + 2y = 6$;
$[f_{\min} = 36/13$ v bode $(18/13; 12/13)]$
d) $f(x, y) = x + y$, $g(x, y) \equiv x^{-2} + y^{-2} - 1 = 0$;
$[f_{\min} = 2\sqrt{2}$ v bode $(\sqrt{2}; \sqrt{2})$ a $f_{\max} = -2\sqrt{2}$ v bode $(-\sqrt{2}; -\sqrt{2})]$
e) $f(x, y) = x^3 + y^3$, $g(x, y) \equiv x + y - 3 = 0$.
$[f_{\min} = 27/4$ v bode $(3/2; 3/2)]$

15. Nájdite $\operatorname{grad} f$ v bode M, ak
a) $f(x, y) = x^3 + y^3 - 3xy$, $M = (2; 1)$; $[9\boldsymbol{i} - 3\boldsymbol{j}]$
b) $f(x, y) = \sqrt{x^2 - y^2}$, $M = (5; 3)$; $[(5\boldsymbol{i} - 3\boldsymbol{j})/4]$
c) $f(x, y, z) = xyz$, $M = (1; 2; 3)$; $[6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}]$
d) $f(x, y, z) = x^2 + y^2 + z^2$, $M = (2; -2; 1)$. $[4\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}]$

16. Nájdite najväčšiu rýchlosť rastu poľa f v bode A:
a) $f(x, y, z) = \ln(x^2 + 4y^2)$, $A = (6; 4; 0)$; $[\sqrt{73}/25]$
b) $f(x, y, z) = x^y - z$, $A = (2; 2; 4)$. $[\sqrt{17 + 16\ln^2 2}]$

17. Nájdite divergenciu a rotáciu vektorových polí:
a) $\boldsymbol{f}(X) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$; $[\operatorname{div} \boldsymbol{f} = 3, \operatorname{rot} \boldsymbol{f} = \boldsymbol{0}]$
b) $\boldsymbol{f}(X) = (x^2 + yz)\boldsymbol{i} + (y^2 + xz)\boldsymbol{j} + (z^2 + xy)\boldsymbol{k}$;

[div $\boldsymbol{f} = 2(x+y+z)$, rot $\boldsymbol{f} = \boldsymbol{0}$]

c) $\boldsymbol{f}(X) = x^2yz\boldsymbol{i} + xy^2z\boldsymbol{j} + xyz^2\boldsymbol{k}$;

[div $\boldsymbol{f} = 6xyz$, rot $\boldsymbol{f} = x(z^2-y^2)\boldsymbol{i} + y(x^2-z^2)\boldsymbol{j} + z(y^2-x^2)\boldsymbol{k}$]

d) $\boldsymbol{f}(X) = e^{xy}\boldsymbol{i} + \cos(xy)\boldsymbol{j} + \cos(xz^2)\boldsymbol{k}$;

[div $\boldsymbol{f} = ye^{xy} - x\sin(xy) - 2xz\sin(xz^2)$,]

[rot $\boldsymbol{f} = z^2\sin(xz^2)\boldsymbol{j} - (xe^{xy} + y\sin(xy))\boldsymbol{k}$]

e) $\boldsymbol{f}(X) = \operatorname{grad}(x^2+y^2+z^2)$. [div $\boldsymbol{f} = 0$, rot $\boldsymbol{f} = \boldsymbol{0}$]