11.5 Cvičenia 11

1. Nájdite Laplaceov obraz funkcie f, ak:
   a) $f(t) = \sin^2 t$; $[\dfrac{2}{p(p^2+4)}]$
   b) $f(t) = \cos^3 t$; $[\dfrac{p^3+7p}{(p^2+9)(p^2+1)}]$
   c) $f(t) = \sin mt \cos nt$; $[\dfrac{m(p^2+m^2-n^2)}{(p^2+m^2+n^2)^2-4m^2n^2}]$
   d) $f(t) = (t+1)\sin 2t$; $[\dfrac{2p^2+4p+8}{(p^2+4)^2}]$
   e) $f(t) = t(e^t + \cosh t)$; $[\dfrac{2(p^2+p+1)}{(p^2-1)^2}]$
   f) $f(t) = t^2 \cos t$. $[\dfrac{2p^3-6p}{(p^2+1)^3}]$

2. Nájdite Laplaceov obraz funkcie f, ak:
   a) $f(t) = \dfrac{e^t-1}{t}$; $[\ln \dfrac{p}{p-1}]$
   b) $f(t) = \dfrac{1-e^{-t}}{t}$; $[\ln \dfrac{p+1}{p}]$
   c) $f(t) = \dfrac{\sin^2 t}{t}$; $[\dfrac{1}{2} \ln \dfrac{\sqrt{p^2+4}}{p}]$
   d) $f(t) = \dfrac{\cos t - cos2t}{t}$; $[\dfrac{1}{2} \ln \dfrac{p^2+4}{p^2+1}]$
   e) $f(t) = \dfrac{e^t-1-t}{t}$; $[\ln \dfrac{p}{p-1} - \dfrac{1}{p}]$
   f) $f(t) = \dfrac{e^t-e^{-t}}{t}$. $[\ln \dfrac{p+1}{p-1}]$

3. Nájdite Laplaceov obraz funkcie f, ak:
   a) $f(t) = e^{3t}\sin^2 t$; $[\dfrac{1}{2(p-3)} - \dfrac{1}{2}\dfrac{p-3}{(p-3)^2+4}]$
   b) $f(t) = e^{-t}t^3$; $[\dfrac{3!}{(p+1)^4}]$
   c) $f(t) = e^t \sinh t$; $[\dfrac{1}{(p-1)^2-1}]$
   d) $f(t) = te^t \cos t$; $[\dfrac{p^2-2p}{(p^2-2p+2)^2}]$
   e) $f(t) = \cos^2(t-b)\eta(t-b)$; $[\dfrac{e^{-bp}}{2p} + \dfrac{pe^{-bp}}{2(p^2+4)}]$
   f) $f(t) = e^{t-2}\eta(t-2)$. $[\dfrac{e^{-2p}}{p-1}]$

4. Nájdite Laplaceov obraz funkcie f, ak:
   a) $f(t) = 2^{-1}e^{3t}(\sin 5t + \sin t)$; $[\dfrac{1}{2}(\dfrac{5}{(p-3)^2+25} + \dfrac{1}{(p-3)^2+1})]$
   b) $f(t) = e^{-t}\dfrac{\sin 7t \sin 3t}{t}$. $[\dfrac{1}{4}\dfrac{(p-1)^2+100}{(p-1)^2+16}]$

5. Nájdite predmet f, ak je daný jeho Laplaceov obraz:
   a) $\dfrac{4p+3}{(p-1)(p^2+2p+5)}$; $\quad[(7e^t - e^{-t}(7\cos 2t - 9\sin 2t))/8]$
   b) $\dfrac{1}{(p+1)^3}(p+3)$; $\quad[(2t^2e^{-t} - 2te^{-t} + e^t - e^{-3t}9/8]$
   c) $\dfrac{e^{-p}}{(p^2-1)p^2}$; $\quad[(\sinh(t-1) - (t-1))\eta(t-1)]$
   d) $\dfrac{p}{(p+1)(p^2+4)}$; $\quad[(2\sin 2t + \cos 2t - e^{-t})/5]$
   e) $\dfrac{p}{(p^2+1)(p^2+9)}$. $\quad[(\cos t - \cos 3t)/8]$

6. Nájdite predmet f, ak je daný jeho Laplaceov obraz:
   a) $\dfrac{1}{p^2+4p+5}$; $\quad[e^{-2t}\sin t]$
   b) $\dfrac{p^2+2p-1}{p^3+3p^2+3p+1}$; $\quad[e^{-t}(1-t^2)]$
   c) $\dfrac{2p+3}{p^3+4p^2+5p}$; $\quad[3/5 + e^{-2t}(4\sin t - 3\cos t)/5]$
   d) $\dfrac{e^{-3p}}{(p+1)^2}$; $\quad[(t-3)e^{3-t}\eta(t-3)]$
   e) $\dfrac{e^{-p}}{p^2-1} + \dfrac{pe^{-2p}}{p^2-4}$. $\quad[\sinh(t-1)\eta(t-1) + \cosh 2(t-2)\eta(t-2)]$

7. Nájdite riešenie danej diferenciálnej rovnice pri daných začiatočných podmienkach:
   a) $x' + 2x = \sin t$, $x(0) = 0$; $\quad[x(t) = (e^{-2t} - \cos t + 2\sin t)/5]$
   b) $x'' + 3x' = e^t$, $x(0) = 0$, $x'(0) = -1$; $\quad[x(t) = e^t/4 + 5e^{-3t}/12 - 2/3]$
   c) $x'' + 2x' - 3x = e^{-t}$, $x(0) = 0$, $x'(0) = 1$; $\quad[x(t) = (3e^t - e^{-3t} - 2e^{-t})/8]$
   d) $x'' + 2x' = t\sin t$, $x(0) = 0$, $x'(0) = 0$;
   $[x(t) = (2e^{-2t} - 2\cos t + 14\sin t - 5t\sin t - 10t\cos t)/25]$
   e) $x''' + x'' = \sin t$, $x(0) = x'(0) = 1$, $x''(0) = 0$;
   $[x(t) = 2t + (e^{-t} + \cos t - \sin t)/2]$
   f) $x'' - x' = te^t$, $x(0) = 0$, $x'(0) = 0$. $\quad[x(t) = e^t(1 - t + t^2/2) - 1]$

8. Nájdite riešenie danej diferenciálnej rovnice pri daných začiatočných podmienkach:
   a) $x'' - 2x' + x = t - \sin t$, $x(0) = 0$, $x'(0) = 0$;
   $[x(t) = 2 + t - (\cos t + te^t - 3e^t)/2]$
   b) $x'' + x = t\cos t$, $x(0) = 0$, $x'(0) = 0$; $\quad[x(t) = (t^2\sin t + t\cos t - \sin t)/4]$
   c) $x'' + 4x = \eta(t) - \eta(t-1)$, $x(0) = 0$, $x'(0) = 0$;
   $[x(t) = [\sin^2 t\eta(t) - \sin(t-1)\eta(t-1)]/2]$
   d) $x'' + x' = 4\sin^2 t$, $x(0) = 0$, $x'(0) = -1$;
   $[x(t) = 2t - 3 + 3e^{-t} - (\sin 2t - 2\cos 2t + 2e^{-t})/5]$
   e) $x'' - x' = t^2$, $x(0) = 0$, $x'(0) = 1$; $\quad[x(t) = 3e^t - 3 - 2t - t^2 - t^3/3]$
   f) $x'' + 4x = \sin t$, $x(0) = 0$, $x'(0) = 0$. $\quad[x(t) = (2\sin t - \sin 2t)/6]$

9. Nájdite riešenie danej diferenciálnej rovnice pri daných začiatočných podmienkach:
   a) $x'' + x = \begin{cases} 0 & \text{pre } t < 0 \text{ a } t > 2, \\ 1 & \text{pre } t \in (0,1), \\ -1 & \text{pre } t \in (1,2), \end{cases}$, $x(0) = 0$, $x'(0) = 0$;
   $[x(t) = 2[\sin^2 \tfrac{t}{2}\eta(t) - 2\sin^2 \tfrac{t-1}{2}\eta(t-1) + \sin^2 \tfrac{t-2}{2}\eta(t-2)]]$

b) $x'' + 4x = \begin{cases} 0 & \text{pre } t < 0 \text{ a } t > 2, \\ t & \text{pre } t \in (0,1), \\ 1-t & \text{pre } t \in (1,2), \end{cases}$, $x(0) = 0$, $x'(0) = 0$;

$[x(t) = 2[\sin^2 \frac{t}{2}\eta(t) - 2\sin^2 \frac{t-1}{2}\eta(t-1) + \sin^2 \frac{t-2}{2}\eta(t-2)]]$

c) $x'' + x = \begin{cases} 0 & \text{pre } t < 0, \\ 1 & \text{pre } t \in (0,1), \\ 2 & \text{pre } t \in (1,\infty), \end{cases}$, $x(0) = 1$, $x'(0) = 0$;

$[x(t) = \eta(t) + [1 - \cos(t-1)]\eta(t-1)]$

d) $x'' + x = \begin{cases} 0 & \text{pre } t < 1 \text{ a } t > 3, \\ t-1 & \text{pre } t \in (1,2), \\ 3-t & \text{pre } t \in (2,3), \end{cases}$, $x(0) = 0$, $x'(0) = 1$.

$[x(t) = \frac{1}{3}\sin 3t\eta(t) + \frac{1}{9}[(t-1) - \frac{1}{3}\sin 3(t-1)]\eta(t-1)] - \frac{2}{9}[(t-2) - \frac{1}{3}\sin 3(t-$ $2)]\eta(t-2)] + \frac{1}{9}[(t-3) - \frac{1}{3}\sin 3(t-3)]\eta(t-3)]$

10. Pomocou Duhamelovho integrálu nájdite riešenie danej diferenciálnej rovnice pri daných začiatočných podmienkach:

a) $x'' - x' = \dfrac{e^{2t}}{(1+e^t)^2}$, $x(0) = 0$, $x'(0) = 0$;

$[x(t) = (e^t - 1)/2 + \ln 2 - \ln(1 + e^t)]$

b) $x'' + 2x' + x = \dfrac{e^{-t}}{1+t}$, $x(0) = 0$, $x'(0) = 0$;

$[x(t) = e^{-t[(t+1)\ln(t+1)-t]}]$

c) $x'' - x' = \dfrac{e^{2t}}{2+e^t}$, $x(0) = 0$, $x'(0) = 0$.

$[x(t) = (e^t + 2)[\ln(e^t + 2) - \ln 3 - e^t + 1]]$

11. Nájdite riešenie danej sústavy diferenciálnych rovníc pri daných začiatočných podmienkach:

a) $\begin{cases} x' + y = 0, \\ y' + x = 0, \end{cases}$, $x(0) = 1$, $y(0) = -1$; $[x(t) = e^t,\ y(t) = -e^t]$

b) $\begin{cases} x' + x = y + e^t, \\ y' + y = x + e^t, \end{cases}$, $x(0) = 1$, $y(0) = 1$; $[x(t) = e^t,\ y(t) = e^t]$

c) $\begin{cases} x' = -y, \\ y' = 2x + 2y, \end{cases}$, $x(0) = 1$, $y(0) = 1$;

$[x(t) = e^t(\cos t - 2\sin t),\ y(t) = e^t(\cos t + 3\sin t)]$

d) $\begin{cases} x' = 3y - x, \\ y' = x + y + e^{at}, \end{cases}$, $x(0) = 1$, $y(0) = 1$;

$$[x(t) = \frac{3e^{-2t}}{4(2+a)} + \frac{(11-4a)e^{2t}}{4(2-a)} + \frac{3e^{at}}{a^2-4},$$
$$y(t) = -\frac{e^{-2t}}{4(2+a)} + \frac{(11-4a)e^{2t}}{4(2-a)} + \frac{(a+1)e^{at}}{a^2-4}]$$

e) $\begin{cases} x' = -y - z, \\ y' = -x - z, \\ z' = -x - y, \end{cases}$, $x(0) = -1$, $y(0) = 0$, $z(0) = 1$;

$[x(t) = e^{-t},\ y(t) = 0,\ z(t) = e^t]$

f) $\begin{cases} x' = 2x - y + z, \\ y' = x + z, \\ z' = -3x + y - 2z, \end{cases}$, $x(0) = 1$, $y(0) = 1$, $z(0) = 0$;

$[x(t) = 2 - e^{-t},\ y(t) = 2 - e^{-t},\ z(t) = 2e^{-t} - 2]$

g) $\begin{cases} x' = -2x - 2y - 4z, \\ y' = -2x + y - 2z, \\ z' = 5x + 2y + 7z, \end{cases}$, $x(0) = 1$, $y(0) = 1$, $z(0) = 1$.

$[x(t) = 6e^t - e^{2t} - 4e^{3t},\ y(t) = 3e^t - 2e^{3t},\ z(t) = 6e^{3t} + e^{2t} - 6e^t]$